韓國精密工學會 2005 年度 秋季學術大會論文集

스펙트럴 요소 모델을 이용한 스펙트럴 해석법

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A SPECTRAL ANALYSIS METHOD FOR SPECTRAL ELEMENT MODELS

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ABSTRACT

In the literatures, the FFT-based SAM has been well applied to the computation of the steady-state responses of discrete dynamic systems. In this paper, a fast Fourier transforms (FFT)-based spectral analysis method (SAM) is proposed for the dynamic analysis of spectral element models subjected to the non-zero initial conditions. However, the FFT-based SAM has not yet been developed for the continuous systems represented by the spectral element model.

Key Words : Spectral Element Model(스펙트럴요소모델), Fast Fourier Transform(고속푸리에변환), Dynamic Response(동적응답)

1. INTRODUCTION

By virtue of impressive progress in computer technologies during last three decades, there have been developed diverse computer-based numerical methods to obtain satisfactory approximate solutions for the discrete dynamic systems with a large degrees of freedom (DOFs). They may include various direct integration methods, the modal analysis methods, the discrete-time system methods, and the spectral analysis methods in which the FFT techniques are utilized. The first three are the time-domain methods [1, 2], while the FFT-based spectral analysis method (SAM) is a frequency-domain method [3-7].

In the FFT-based SAM for discrete dynamic systems, the dependent variables of a set of ordinary differential equations are transformed into the frequency-domain by using the discrete Fourier transforms (DFT) to transform the ordinary differential equations into a set of algebraic equations with frequency as a parameter. The algebraic equations are then solved for the Fourier (or spectral) components of dependent variables at each discrete frequency. As the final step, the time-domain responses are reconstructed from the Fourier components by using the inverse discrete Fourier transforms (IDFT). In practice, the FFT is used to conduct the DFT or IDFT.

As the FFT is a remarkably efficient computer algorithm, it cannot only offer an enormous reduction in computer time but also increase the accuracy of solutions [4-6]. The FFT-based SAM has been known to be very useful especially in the following situations [4-7]: (1) when the modern data acquisition systems are used, as in most experimental measurements, to store digitized data through the analogue-to-digital converters, (2) when the excitation forces are so complicated that one has to use numerical integration to obtain the dynamic responses by using the excitation values at a discrete set of instants, (3) when it is significantly easier to measure the constitutive equation of a material in the frequency-domain rather than in the time-domain, and (4) when the frequency-dependent spectral finite element (or dynamic stiffness matrix) model is used for a structure.

In the literatures [3-7], the FFT-based SAM has been well applied to the computation of the steady-state responses of discrete dynamic systems. The applications of the FFT-based SAM to the transient responses of dynamic systems have been limited to the cases when all initial conditions are zero. To take into account the nonzero initial conditions, Humar and Xia [9] and Veletsos and Ventura [10] introduced the DFT-based procedures for calculating

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the transient response of a linear one DOF system from its corresponding steady-state response to a periodic extension of the excitation. The procedure involves the superposition of a corrective, free vibration solution which effectively transforms the steady-state response to the desired transient response. Mansur et al. [12, 13] used the pseudo-force concept to take into account the non-zero initial conditions in the DFT-based frequency-domain analysis of continuous media discretized by the FEM. The reference [12] solved the dynamic problem in the modal coordinates, whereas the reference [13] in both nodal and modal coordinates. Recently Lee at al. [14] and Cho and Lee [15] developed the FFT-based SAMs for the linear discrete dynamic systems subjected to non-zero initial conditions. To the authors' best knowledge, the FFT-based SAM has not yet been developed for the continuous systems represented by the spectral element model. The readers may refer to references [4-6] for the spectral element models.

Thus, the purpose of this paper is to develop an FFT-based SAM for the linear spectral element models subjected to non-zero initial conditions. To evaluate the proposed FFT-based SAM, the forced vibration of a simply supported Bernoulli-Euler beam is considered as an illustrative problem.

2. A BRIEF REVIEW ON DFT THEORY

Because the theory of discrete Fourier transforms (DFT) is one of the key mathematical tools used to develop the present FFT-based SAM, a brief review on the DFT theory will be given in the following. A periodic function of time $x(t)$, with the period T, can be always expressed as a Fourier series of the form

$$x(t) = a_0 + 2\sum_{n=0}^{\infty}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right) = \sum_{n=-\infty}^{\infty} X_n e^{i\omega_n t} \quad (1)$$

where $i = \sqrt{-1}$, ω_n are the discrete frequencies defined by

$$\omega_n = \frac{2\pi}{T} n \equiv \omega_1 n \quad (2)$$

and X_n are constant Fourier components given by

$$X_n = a_n - ib_n = \frac{1}{T}\int_0^T x(t) e^{-i\omega_n t} dt \quad (3)$$

Equations (1) and (3) are the continuous Fourier transforms pair for a periodic function.

Although $x(t)$ is a continuous function of time, it is often the case that only sampled values of the function are available, in the form of a discrete time series $\{x(t_r)\}$. If N is the number of samples, all equally spaced with a time interval equal to $\Delta = T/N$, the discrete time series are given by $x_r = x(t_r)$, where $t_r = r\Delta$ and $r = 0, 1, 2, \ldots, N-1$. The integral in Eq. (3) may be replaced approximately by the summation

$$X_n = \sum_{r=0}^{N-1} x(t_r) e^{-i\omega_n t_r} \quad (n = 0, 1, 2, \ldots, N\text{-}1) \quad (4)$$

which is the discrete Fourier transforms (DFT) of the discrete time series $\{x_r\}$. Any typical value x_r of the series $\{x_r\}$ can be given by the inverse formula

$$x(t_r) = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{i\omega_n t_r} \quad (r = 0, 1, 2, \ldots, N\text{-}1) \quad (5)$$

which is the inverse discrete Fourier transforms (IDFT). Thus, Eq. (4) and Eq. (5) represent the DFT-IFFT fair. Even though Eq. (4) is an approximation of Eq. (3), it is important to note that it allows all discrete time series $\{x_r\}$ to be regained exactly [7, 8]. The Fourier components Xn in Eq. (5) are arranged as $X_{N-n} = X_n^*$ ($n = 0, 1, 2, \ldots, N/2$), where X_n^* represents the complex conjugate of X_n. Note that $X_{N/2}$ corresponds to the highest frequency $\omega_{N/2} = (N/2)\omega_1$, which is called the Nyquist frequency.

The fast Fourier transforms (FFT) is an ingenious highly efficient computer algorithm developed to perform the numerical operations required for a DFT or IDFT, reducing the computing time drastically by the order $N/\log 2N$. It should be pointed out that while the FFT-based spectral analysis uses a computer, it is not a numerical method in the usually sense, because the analytical descriptions of Eqs. (4) and (5) are still retained. Further details of DFT and FFT can be found in the reference [8].

3. SPECTRAL ELEMENT MODEL

In the frequency-domain, the dynamics of a linear continuous structural system can be represented by the spectral element model or exact dynamic stiffness matrix model which is expressed in the matrix form as

$$\boldsymbol{S}(\omega)\overline{\boldsymbol{D}} = \overline{\boldsymbol{F}} \quad (6)$$

where $\overline{\boldsymbol{F}}$ represents the magnitudes of the harmonic forces applied at the nodal points and $\overline{\boldsymbol{D}}$ represents the magnitudes of the corresponding frequency responses. The matrix $\boldsymbol{S}(\omega)$ is the exact dynamic stiffness matrix for a whole system, which can be derived by assuming the harmonic solutions at a circular frequency ω as discussed in the following.

As an example, consider the vibration of a uniform Bernoulli-Euler beam represented by

$$EIw''''(x,t) + \rho A\ddot{w}(x,t) = 0 \quad (7)$$

where $w(x,t)$ is the transverse displacement, E is the Young's modulus, A is the cross-sectional area, I is the area moment of inertia, and ρ is the mass density. In Eq. (7), the prime ($'$) and the dot ($\cdot$) indicate the derivatives with respect to the axial coordinate *x* and the time *t*, respectively. The bending moment *M*(*x*, *t*) and the transverse shear force *V*(*x*, *t*) are related to the displacement field as

$$M(x,t) = EIw''(x,t) \\ V(x,t) = -EIw'''(x,t) \tag{8}$$

Assume the harmonic solution for Eq. (7) as

$$w(x,t) = \bar{w}(x)e^{i\omega t} \tag{9}$$

Substitute Eq. (9) into Eq. (7) to obtain an ordinary differential equation as

$$EI\bar{w}''''(x) - \rho A\omega^2\bar{w}(x) = 0 \tag{10}$$

Assume the general solution of Eq. (10) as

$$\bar{w}(x) = a e^{kx} \tag{11}$$

where *k* is the wavenumber and a is the constant coefficient. Substituting Eq. (11) into Eq. (10) yields a characteristic equation as

$$k^4 - \beta^4 = 0 \tag{12}$$

where

$$\beta = \omega\sqrt{\frac{\rho A}{EI}} \tag{13}$$

From Eq. (12), we can compute four roots (i.e., wavenumbers) as

$$k_1 = -k_2 = \beta\ ,\ k_3 = -k_4 = i\beta \tag{14}$$

For a finite beam element of length l, the general solution of Eq. (10) can be then expressed as

$$\bar{w}(x) = \sum_{j=1}^{4} a_j e^{k_j x} = \boldsymbol{G}(x)\boldsymbol{A} \tag{15}$$

where

$$\boldsymbol{G}(x) = \begin{bmatrix} e^{k_1 x} & e^{k_2 x} & e^{k_3 x} & e^{k_4 x} \end{bmatrix} \\ \boldsymbol{A} = \{a_1 \quad a_2 \quad a_3 \quad a_4\}^{\mathrm{T}} \tag{16}$$

The displacements and slopes at the nodal points of the finite beam element can be represented as

$$\bar{\boldsymbol{d}} = \begin{Bmatrix} \bar{w}_1 \\ \bar{w}_1' \\ \bar{w}_2 \\ \bar{w}_2' \end{Bmatrix} = \begin{Bmatrix} \bar{w}(x=0) \\ \bar{w}'(x=0) \\ \bar{w}(x=l) \\ \bar{w}'(x=l) \end{Bmatrix} \tag{17}$$

Substituting Eq. (15) into Eq. (17) gives

$$\bar{\boldsymbol{d}} = \begin{bmatrix} \boldsymbol{G}(0) \\ \boldsymbol{G}'(0) \\ \boldsymbol{G}(l) \\ \boldsymbol{G}(l) \end{bmatrix} \boldsymbol{A} \equiv \boldsymbol{Q}(\omega)\boldsymbol{A} \tag{18}$$

From Eq. (8), the magnitudes of harmonic bending moment and transverse shear force can be obtained from

$$\bar{M}(x) = EI\bar{w}''(x)\ ,\ \bar{V}(x) = -EI\bar{w}'''(x) \tag{19}$$

Define the bending moments and transverse shear forces at nodal points as follows

$$\bar{\boldsymbol{f}} = \begin{Bmatrix} \bar{V}_1 \\ \bar{M}_1 \\ \bar{V}_2 \\ \bar{M}_2 \end{Bmatrix} = \begin{Bmatrix} -\bar{V}(0) \\ -\bar{M}(0) \\ \bar{V}(l) \\ \bar{M}(l) \end{Bmatrix} \tag{20}$$

Substituting Eqs. (15) and (19) into Eq. (20) gives

$$\bar{\boldsymbol{f}} = EI\begin{bmatrix} \boldsymbol{G}'''(0) \\ -\boldsymbol{G}''(0) \\ -\boldsymbol{G}''(l) \\ \boldsymbol{G}''(l) \end{bmatrix} \boldsymbol{A} \equiv \boldsymbol{R}(\omega)\boldsymbol{A} \tag{21}$$

Eliminating the constants vector A from Eq. (18) and Eq. (21) gives

$$\bar{\boldsymbol{f}} = \boldsymbol{R}(\omega)\boldsymbol{Q}(\omega)^{-1}\bar{\boldsymbol{d}} \equiv \boldsymbol{s}(\omega)\bar{\boldsymbol{d}} \tag{22}$$

where $\boldsymbol{s}(\omega)$ is the frequency-dependent dynamic stiffness matrix for the finite Bernoulli-Euler beam element of length l. If a structure consists of many finite elements, each finite elements represented by Eq. (22) can be assembled in a completely analogous way used in the conventional FEM to derive a global system equation in the form of Eq. (6).

Since Eq. (6) is valid at any harmonic frequency, one may select the harmonic frequencies to be the discrete frequencies defined by Eq. (2). Then, at the nth discrete frequency ωn, for instance, Eq. (6) satisfies

$$\boldsymbol{S}(\omega_n)\bar{\boldsymbol{D}}_n = \bar{\boldsymbol{F}}_n \quad \text{or} \quad \boldsymbol{S}_n\bar{\boldsymbol{D}}_n = \bar{\boldsymbol{F}}_n \tag{23}$$

In the literatures [4-6], $\boldsymbol{S}_n$, $\bar{\boldsymbol{D}}_n$ and $\bar{\boldsymbol{F}}_n$ defined at discrete frequencies are often called the (assembled) spectral element matrix, the spectral nodal DOFs vector, and the spectral nodal forces vector, respectively.

4. DYNAMIC RESPONSE

The time-domain dynamic response $\boldsymbol{D}(t)$ of Eq. (23) can be obtained by the sum of the steady–state response (particular solution) $\boldsymbol{P}(t)$ and the transient response (homogenous solution) $\boldsymbol{H}(t)$.

$$\boldsymbol{D}(t) = \boldsymbol{P}(t) + \boldsymbol{H}(t) \tag{24}$$

By the use of DFT theory, the time-domain solution $\boldsymbol{D}(t)$ can be readily reconstructed from its spectra $\bar{\boldsymbol{D}}_n$ (n = 0, 1, …, N-1) as follows:

$$\boldsymbol{D}(t) = \frac{1}{N}\sum_{n=0}^{N-1}\bar{\boldsymbol{D}}_n e^{i\omega_n t} \tag{25}$$

Hence, the goal here is to develop a methodology to compute $\bar{\boldsymbol{D}}_n$.

Assume that $\boldsymbol{P}(t)$ and $\boldsymbol{H}(t)$ can be expressed in the spectral forms as

$$\boldsymbol{P}(t) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\boldsymbol{P}}_n\, e^{i\omega_n t}$$
$$\boldsymbol{H}(t) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n\, e^{i\omega_n t} \tag{26}$$

From Eqs. (24-26), one can readily show the relation as

$$\overline{\boldsymbol{D}}_n = \overline{\boldsymbol{P}}_n + \overline{\boldsymbol{H}}_n \tag{27}$$

4.1 Computation of steady-state response part

For the given spectral nodal force vector $\overline{\boldsymbol{F}}_n$ (n = 0, 1, 2,…, N-1), the spectra of particular solution part, $\overline{\boldsymbol{P}}_n$, should satisfy

$$\boldsymbol{S}_n \overline{\boldsymbol{P}}_n = \overline{\boldsymbol{F}}_n \tag{28}$$

From Eq. (28), one can compute

$$\overline{\boldsymbol{P}}_n = \boldsymbol{S}_n^{-1} \overline{\boldsymbol{F}}_n, \quad \boldsymbol{P}_{N-n} = \boldsymbol{P}_n^{*} \qquad (n = 1, 2, \ \ldots , N/2) \tag{29}$$

where the symbol (*) indicates the complex conjugate. The steady–state response in the time-domain can be reconstructed, by using FFT algorithm, from $\overline{\boldsymbol{P}}_n$ (n = 0, 1, 2,…, N-1) as follows:

$$\boldsymbol{P}(t) \Leftarrow IFFT\left\{\overline{\boldsymbol{P}}_n\right\} \tag{30}$$

From Eq. (26), the time derivative of $\boldsymbol{P}(t)$ can be obtained as

$$\dot{\boldsymbol{P}}(t) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\dot{\boldsymbol{P}}}_n\, e^{i\omega_n t} \tag{31}$$

where

$$\overline{\dot{\boldsymbol{P}}}_n = (i\,\omega_n)\overline{\boldsymbol{P}}_n, \quad \overline{\dot{\boldsymbol{P}}}_{N-n} = \overline{\dot{\boldsymbol{P}}}_n^{*} \qquad (n = 1, 2, \ldots , N/2) \tag{32}$$

4.2 Computation of transient response part

Without the external forces, Eq. (6) becomes

$$\boldsymbol{S}(\omega)\overline{\boldsymbol{X}} = \boldsymbol{0} \tag{33}$$

which is an eigenvalue problem. One can compute the complex eigenvalues Ω_j (j = 1, 2,…, ∞) from

$$\det \boldsymbol{S}(\Omega_j) = 0 \tag{34}$$

and the corresponding eigenvectors $\overline{\boldsymbol{X}}_j$ (j = 1, 2,…, ∞). Then the time-domain transient response $\boldsymbol{H}(t)$ can be written as

$$\boldsymbol{H}(t) = \sum_{j=1}^{\infty} b_j \overline{\boldsymbol{X}}_j\, e^{i\Omega_j t} \cong \sum_{j=1}^{M} b_j\, \overline{\boldsymbol{X}}_j\, e^{i\Omega_j t} \tag{35}$$

where bj are constants. The time derivative of $\boldsymbol{H}(t)$ can be obtained from Eq. (35) as

$$\dot{\boldsymbol{H}}(t) \cong \sum_{j=1}^{M} b_j\,(i\Omega_j)\overline{\boldsymbol{X}}_j\, e^{i\Omega_j t} \tag{36}$$

Represent $\boldsymbol{H}(t)$ and $\dot{\boldsymbol{H}}(t)$ into the spectral forms as

$$\boldsymbol{H}(t_r) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\boldsymbol{H}}_n\, e^{i\omega_n t_r}$$
$$\dot{\boldsymbol{H}}(t_r) = \frac{1}{N}\sum_{n=0}^{N-1} \overline{\dot{\boldsymbol{H}}}_n\, e^{i\omega_n t_r} \tag{37}$$

where, by the DFT theory, the Fourier spectra $\boldsymbol{H}_n$ and $\dot{\boldsymbol{H}}_n$ can be computed from

$$\overline{\boldsymbol{H}}_n = \sum_{r=0}^{N-1} \boldsymbol{H}(t_r)e^{-i\omega_n t_r}$$
$$\overline{\dot{\boldsymbol{H}}}_n = \sum_{r=0}^{N-1} \dot{\boldsymbol{H}}(t_r)e^{-i\omega_n t_r} \tag{38}$$

Substituting Eq. (35) and Eq. (36) into Eq. (38) gives

$$\overline{\boldsymbol{H}}_n = \sum_{j=1}^{M} b_j\, \boldsymbol{L}_{jn}$$
$$\overline{\dot{\boldsymbol{H}}}_n = \sum_{j=1}^{M} b_j\,(i\boldsymbol{\Omega}_j)\boldsymbol{L}_{jn} \tag{39}$$

where,

$$\boldsymbol{L}_{jn} = \overline{\boldsymbol{X}}_j\, \frac{1-e^{i(\boldsymbol{\Omega}_j-\omega_n)\Delta N}}{1-e^{i(\boldsymbol{\Omega}_j-\omega_n)\Delta}} \tag{40}$$

The constants b_j appeared in Eq. (35), Eq. (36), and Eq. (39) will be determined to satisfy the initial conditions given by

$$\boldsymbol{D}(0) = \boldsymbol{D}_0 \ , \ \dot{\boldsymbol{D}}(0) = \dot{\boldsymbol{D}}_0 \tag{41}$$

The total dynamic response determined from Eq. (24), should satisfy the initial conditions, Eq. (41). Thus, applying the initial conditions Eq. (41) into Eq. (24) gives

$$\boldsymbol{D}_0 = \boldsymbol{P}(0) + \boldsymbol{H}(0)$$
$$\dot{\boldsymbol{D}}_0 = \dot{\boldsymbol{P}}(0) + \dot{\boldsymbol{H}}(0) \tag{42}$$

Applying Eqs. (26a), (31) and (37) into Eq. (42) gives

$$\boldsymbol{D}_0 = \frac{1}{N}\sum_{n=0}^{N-1}\left(\overline{\boldsymbol{P}}_n + \overline{\boldsymbol{H}}_n\right)$$
$$\dot{\boldsymbol{D}}_0 = \frac{1}{N}\sum_{n=0}^{N-1}\left(\overline{\dot{\boldsymbol{P}}}_n + \overline{\dot{\boldsymbol{H}}}_n\right) \tag{43}$$

Substituting Eq. (39) into Eq. (43) gives

$$\sum_{n=0}^{N-1}\sum_{j=1}^{M} b_j\, \boldsymbol{L}_{jn} = N\boldsymbol{D}_0 - \sum_{n=0}^{N-1} \overline{\boldsymbol{P}}_n \equiv \boldsymbol{p}$$
$$\sum_{n=0}^{N-1}\sum_{j=1}^{M} b_j\,(i\boldsymbol{\Omega}_j)\boldsymbol{L}_{jn} = N\dot{\boldsymbol{D}}_0 - \sum_{n=0}^{N-1} \overline{\dot{\boldsymbol{P}}}_n \equiv \boldsymbol{v} \tag{44}$$

Eq. (44) can be represented as

$$\sum_{j=1}^{M} b_j \boldsymbol{Z}_j = N\boldsymbol{D}_0 - \sum_{n=0}^{N-1} \overline{\boldsymbol{P}}_n \equiv \boldsymbol{p}$$
$$\sum_{j=1}^{M} b_j \left(i\boldsymbol{\Omega}_j\right)\boldsymbol{Z}_j = N\dot{\boldsymbol{D}}_0 - \sum_{n=0}^{N-1} \dot{\overline{\boldsymbol{P}}}_n \equiv \boldsymbol{v} \tag{45}$$

where

$$\boldsymbol{Z}_j = \sum_{n=0}^{N-1} \boldsymbol{L}_{jn} \tag{46}$$

Equation (45) can be rewritten as

$$\begin{bmatrix} \boldsymbol{R}_0 \\ \boldsymbol{R}_1 \end{bmatrix} \boldsymbol{B} = \begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{Bmatrix} \tag{47}$$

where

$$\boldsymbol{B} = \{ b_1 \quad b_2 \quad \cdots \quad b_M \}$$
$$\boldsymbol{R}_k = \left[(i\boldsymbol{\Omega}_1)^k \boldsymbol{Z}_1 \quad (i\boldsymbol{\Omega}_2)^k \boldsymbol{Z}_2 \quad \cdots \quad (i\boldsymbol{\Omega}_M)^k \boldsymbol{Z}_M \right] \tag{48}$$
$$(k = 0,1)$$

Since the vectors p and v in the right of Eq. (47) can be computed in advance from the initial values $\boldsymbol{D_0}$ and $\dot{\boldsymbol{D}}_0$ and the steady-state responses $\overline{\boldsymbol{P}}_n$ and $\dot{\overline{\boldsymbol{P}}}_n$, the unknown constants vector $\boldsymbol{B}$ can be solved from Eq. (47) by using a proper solution approach. (The solution approach will be further detailed in the final manuscript.) Once $\boldsymbol{B}$ is solved from Eq. (48), the transient response part can be then computed from Eq. (47), by using FFT, through Eq. (39).

ACKNOWLEDGEMENTS

This work was supported by the Korea Research Foundation Grant (KRF-2004-041-D00034)

APPENDIX

The spectral element matrix of the Bernoulli-Euler beam, $\boldsymbol{s}(\omega)$ is

$$\boldsymbol{s}(\omega) = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{22} & s_{23} & s_{24} \\ s_{13} & s_{23} & s_{33} & s_{34} \\ s_{14} & s_{24} & s_{34} & s_{44} \end{bmatrix} = \boldsymbol{s}^{\mathrm{T}}(\omega) \tag{a}$$

where

$$\begin{aligned} s_{11} = s_{33} &= \alpha L^2 \beta^2 [\cos(\beta L)\sinh(\beta L) \\ &\quad + \sin(\beta L)\cosh(\beta L)] \\ s_{22} = s_{44} &= \alpha L^2 [-\cos(\beta L)\sinh(\beta L) \\ &\quad + \sin(\beta L)\cosh(\beta L)] \\ s_{12} = -s_{34} &= \alpha L^2 \beta \sin(\beta L)\sinh(\beta L) \\ s_{13} &= \alpha L^2 \beta^2 [-\sin(\beta L) - \sinh(\beta L)] \\ s_{14} = -s_{23} &= \alpha L^2 \beta [-\cos(\beta L) + \cosh(\beta L)] \\ s_{24} &= \alpha L^2 [-\sin(\beta L) + \sinh(\beta L)] \end{aligned} \tag{b}$$

with

$$\alpha = \frac{EI}{L^3} \left(\frac{\beta L}{1 - \cos(\beta L)\cosh(\beta L)} \right) \tag{c}$$

and L is the length of beam.

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