쇼케이적분과 퍼지측도

Choquet integrals and fuzzy measures

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요 약

이 논문발표에서는 퍼지측도와 쇼케이적분을 소개하고 지금까지 나온 결과들과 앞으로 가능한 응용들에 대해서 소개하고자한다.

Abstract

In this paper, we consider fuzzy measures and Choquet integrals. Also we discuss some results proved by us and new works.

Key Words : fuzzy measures, Choquet integrals, comonotonically additive functionals, Hausdorff metric.

1. Introduction.

It was well-known that closed set-valued functions had been used repeatedly in many papers [1,2,4-13, 26,27]. Using these properties, we have been studied some characterizations of closed set-valued Choquet integrals ([5,6]) and convergence therems for interval-valued Choquet integrals ([7-9]). We defined Choquet integralof measurable fuzzy number-valued functions and comonotonically additive interval-valued functional which generalize the concept of a comonotonically additive functionals and study some properties of them. And we also investigate some relations between comonotonically additive interval-valued functionals and interval-valued Choquet integrals under sufficient conditions([10-13]).

In section 2, we list various definitions and notations which are used in the proof of our results. In section 3-7, using these definitions and properties, we introduce various results as follows;

(1) Functionals represented by interval-valued Choquet integrals and define comonotonically additive interval-valued functional on a suitable class of interval number-valued functions,

(2) The Choquet expected value of fuzzy number-valued random variables,

(3) The Choquet expected fuzzy number-valued utility functions.

We now study the following topics;

(4) Interval-valued Choquet integrals in multicriteria decision aid,

(5) Order invariant interval-valued aggregation functionals,

(6) Correlation coefficients of interval-valued fuzzy numbers by utilizing Choquet integral instead of Riemann integral, etc.

2. Definitions and preliminaries.

A fuzzy measure on a measurable space (X, Ω) is an extended real-valued function $\mu : \Omega \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \le \mu(B)$, whenever $A, B \in \Omega$, $A \subset B$.

A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets, we have $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1) < \infty$, we have $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. If μ is both lower semi-continuous and upper semi-continuous, it is said to be continuous. Recall that a function $f : X \to [0, \infty]$ is said to be measurable if $\{x | f(x) > \alpha\} \in \Omega$ for all $\alpha \in (-\infty, \infty)$.

Definition 2.1 (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$(C) \int f d\mu = \int_0^{\infty} \mu(\{x | f(x) > r\}) dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called Choquet integrable if the Choquet integral of f can be defined and its value is finite.

Throughout the paper, R^+ will denote the interval $[0,\infty)$,

$$I(R^+)=\{[a,b]\mid a,b\in R^+ \text{ and } a\le b\}.$$

Then a element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a,b],[c,d]\in I(R^+)$ and $k\in R^+$,

$$[a,b]+[c,d]=[a+c,b+d],$$
$$[a,b]\cdot[c,d]=[a\cdot c,b\cdot d],$$
$$k[a,b]=[ka,kb],$$
$$[a,b]\le[c,d] \text{ if and only if } a\le b \text{ and } c\le d,$$

Then $(I(R^+),d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\}$$

for all $A,B\in I(R^+)$. We note that $[a,b]<[c,d]$ if and only if $[a,b]\le[c,d]$ or $[a,b]\ne[c,d]$. It is easily to show that for $[a,b]$, $[c,d]\in I(R^+)$,

$$d_H([a,b],[c,d])=\max\{|a-c|,|b-d|\}.$$

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F: X\to C(R^+)\setminus\{\emptyset\}$ and an interval number-valued function $F: X\to I(R^+)\setminus\{\emptyset\}$. We denote that $d_H-\lim_{n\to\infty}A_n=A$ if and only if $\lim_{n\to\infty}d_H(A_n,A)=0$, where $A\in I(R^+)$ and $\{A_n\}\subset I(R^+)$.

Definition 2.2 A closed set-valued function F is said to be measurable if for each open set $O\subset R^+$,

$$F^{-1}(O)=\{x\in X\mid F(x)\cap O\ne\emptyset\}\in\Omega.$$

Definition 2.3 Let F be a closed set-valued function. A measurable function $f: X\to R^+$ satisfying

$$f(x)\in F(x) \text{ for all } x\in X$$

is called a measurable selection of F.

We say $f: X\to R^+$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C)\int f\,d\mu<\infty$. We note that "$x\in X$ $\mu-a.e.$" stands for "$x\in X$ μ-almost everywhere". The property $P(x)$ holds for $x\in X$ $\mu-a.e.$ means that there is a measurable set A such that $\mu(A)=0$ and the property $P(x)$ holds for all $x\in A^c$, where A^c is the complement of A.

Definition 2.4 Let f,g be measurable nonnegative functions. We say that f and g are comonotonic, in symbol $f\sim g$ if and only if

$$f(x)<f(x')\Rightarrow g(x)\le g(x') \text{ for all } x,x'\in X.$$

Theorem 2.5 Let f,g,h be measurable functions. Then we have

(1) $f\sim f$,
(2) $f\sim g \Rightarrow g\sim f$,
(3) $f\sim a$ for all $a\in R^+$,
(4) $f\sim g$ and $f\sim h \Rightarrow f\sim(g+h)$.

Theorem 2.6 Let f,g be nonnegative measurable functions.

(1) If $f\le g$, then $(C)\int fd\mu\le(C)\int gd\mu$.

(2) If $f\sim g$ and $a,b\in R^+$, then
$$(C)\int(af+bg)d\mu=a(C)\int fd\mu+b(C)\int gd\mu.$$

Definition 2.7 (1) Let F be a closed set-valued function and $A\in\mathfrak{I}$. The Choquet integral of F on A is defined by

$$(C)\int_A Fd\mu=\{(C)\int_A fd\mu \mid f\in S_c(F)\}$$

where $S_c(F)$ is the family of selections of F, that is,

$$S_c(F)=\{f\mid f \textit{ is measurable} \text{ and } f(x)\in F(x)\ x\in X\ \mu-a.e.\}$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int Fd\mu\ne\emptyset$.

(3) A closed set-valued function F is said to be Choquet integrably bounded if there is a function $g\in L_c^1(\mu)$ such that
$\|F(x)\|=\sup_{r\in F(x)}|r|\le g(x)$ for all $x\in X$.

Instead of $(C)\int_X Fd\mu$, we will write $(C)\int Fd\mu$. Let us discuss some properties of interval-valued Choquet integrals which mean Choquet integrals of measurable interval number-valued functions.

Theorem 2.9([12, 27]) Let μ be a continuous fuzzy measure and F a Choquet integrably bounded set-valued function.

(1) If F is closed set-valued, then $(C)\int f\,d\mu$ is closed.

(2) If F is convex set-valued, then $(C)\int f\,d\mu$ is convex.

(3) If F is interval-valued, i.e. $F(x)=[f^-(x),f^+(x)]$, for all $x\in X$, then
$$(C)\int Fd\mu=[(C)\int f_*d\mu,(C)\int f^*d\mu].$$

We recall that f^*,f_* are Choquet integrable selections of F in [8].

3. Comonotonically additive interval-valued functionals.

We assume that X is a locally compact Hausdorff space and the class Ω_1 of its Borel subsets. Let K^+ the set of continuous nonnegative functions defined on X with compact support.

Definition 3.1 Let ℓ be a real-valued functional on K^+.

(1) ℓ is comonotonically additive if and only if

$$f \sim g \Rightarrow \ell(f+g) = \ell(f) + \ell(g) \text{ for all } f, g \in K^+.$$

(2) ℓ is positively homogeneous if and only if

$$\ell(af) = a\ell(f) \text{ for all } a \in R^+ \text{ and } f \in K^+.$$

(3) ℓ is monotonic if and only if

$$f \le g \Rightarrow \ell(f) \le \ell(g) \text{ for all } f, g \in K^+.$$

Since the Choquet integral with respect to every fuzzy measure is a comonotonically additive, positively homogeneous and monotonic functional, we have the following corollary.

Corollary 3.2 For every fuzzy measure μ, there exists a outer regular measure μ_r such that for every $f \in K^+$,

$$(C)\int f d\mu = (C)\int f d\mu_r.$$

We consider interval-valued Choquet integrals with respect to fuzzy measure and will define comonotonically additive, positively homogeneous and monotonic interval-valued functional on the class $\mathfrak{I}[\mathfrak{I}_1]$ of [Choquet integrably bounded] interval number-valued functions.

Definition 3.3 Let $F, G \in \mathfrak{I}$. We say that F and G are comonotonic, in symbol, $F \sim G$ if and only if

(i) $f^*(x) < f^*(x') \Rightarrow g^*(x) \le g^*(x')$ for all $x, x' \in X$, and
(ii) $f_*(x) < f_*(x') \Rightarrow g_*(x) \le g_*(x')$ for all $x, x' \in X$,
where
$f^*(x) = \sup\{F(x)\}$, $f_*(x) = \inf\{F(x)\}$, $g^*(x) = \sup\{G(x)\}$, and $g_*(x) = \inf\{G(x)\}$.

From Definition 3.3, clearly we have the following theorem.

Theorem 3.4 Let $F, G \in \mathfrak{I}$. Then we have

(1) $F \sim F$,
(2) $F \sim G \Rightarrow G \sim F$,
(3) $F \sim A$ for all $A \in I(R^+)$,
(4) $F \sim G,\ \ F \sim H \Rightarrow F \sim G+H$.

Theorem 3.5 Let $F, G \in \mathfrak{I}_1$. If $F \sim G$, then we have

$$(C)\int (F+G) d\mu = (C)\int F d\mu + (C)\int G d\mu.$$

Theorem 3.6 Let $F, G \in \mathfrak{I}_1$. Then we have

(1) $(C)\int aF d\mu = a(C)\int F d\mu$ for all $a \in R^+$,

(2) if $F \le G$, then $(C)\int F d\mu \le (C)\int G d\mu$.

We consider the class of interval number-valued functions with continuous selections;

$$\mathfrak{I}_2 = \{F \in \mathfrak{I}_1 \mid S_c(F) \subset K^+\}.$$

Definition 3.7 (1) A mapping $T: \mathfrak{I}_2 \to I(R^+)$ is said to be an interval-valued functional.

(2) An interval-valued functional T is comonotonically additive if and only if

$$F \sim G \Rightarrow T(F+G) = T(F) + T(G).$$

(3) An interval-valued functional T is positively homogeneous if and only if

$$T(aF) = aT(F) \text{ for all } a \in R^+.$$

(4) An interval-valued functional T is monotonic if and only if for each pair $F, G \in \mathfrak{I}_2$,

$$F \le G \Rightarrow T(F) \le T(G).$$

Definition 3.8 Let $\ell : K^+ \to R^+$ be a real-valued functional. A mapping $T_\ell : \mathfrak{I}_2 \to I(R^+)$ is said to be an interval-valued functional induced by ℓ if for all $F \in \mathfrak{I}_2$,

$$T_\ell(F) = \{\ell(f) \mid f \in S_c(F)\}.$$

Theorem 3.9 If $T: \mathfrak{I}_2 \to I(R^+)$ is defined by $T(F) = (C)\int F d\mu$ for all $F \in \mathfrak{I}_2$, then T is a comonotonicallly additive, positively homogeneous, monotonic interval-valued functional.

Theorem 3.10 Let $\ell : K^+ \to R^+$ be a comonotonicallly additive, positively homogeneous, monotonic functional. If T_ℓ an interval-valued functional induced by ℓ, there exists a outer regular fuzzy measure μ on Ω_1 such that for all $F \in \mathfrak{I}_2$,

$$T_\ell(F) = (C)\int F d\mu = [\,\ell(f_*),\ \ell(f^*)],$$

where $f^*(x) = \sup\{F(x)\}$ and $f_*(x) = \inf\{F(x)\}$.

Theorem 3.11 Let $\ell : K^+ \to R^+$ be a real-valued functional. If ℓ is comonotonically additive, positively homogeneous and monotonic, so is T_ℓ.

4. The Choquet expected value of fuzzy number-valued random variables.

Definition 4.1 (1) A mapping $\tilde{Y}: \Omega \to F(R^+)$ is called a fuzzy number-valued random variable if for each $\alpha \in [0,1]$, $[\tilde{Y}]^\alpha : \Omega \to I(R^+)$ is an interval number-valued random variable.

(2) $\tilde{Y}$ is called Choquet integrably bounded if $[\tilde{Y}]^0$ is Choquet integrably bounded.

Let $\tilde{Y}: \Omega \to F(R^+)$ be Choquet integrably bounded fuzzy number-valued random variable. We define the Choquet expected value(denoted by $E_c(\tilde{Y})$) of $\tilde{Y}$ as that element $V \in F(R^+)$ which satisfies

$$[V]^\lambda = (C)\int [\tilde{Y}]^\lambda d\mu, \text{ for all } \lambda \in [0,1].$$

We should prove that effectively the family $\{(C)\int [\tilde{Y}]^\lambda d\mu \mid \lambda \in [0,1]\}$ defines a fuzzy set for this purpose, we using the following lemma:

Lemmma 4.2 Let $\{[a^\lambda, b^\lambda] \mid \lambda \in [0,1]\}$ be a given family of nonempty interval numbers. If (i) for all $0 \le \lambda_1 \le \lambda_2 \le 1$, $[a^{\lambda_1}, b^{\lambda_1}] \supset [a^{\lambda_2}, b^{\lambda_2}]$ and (ii) for any nonincreasing sequence $\{\lambda_k\}$ in $[0,1]$ in converging to λ, $[a^\lambda, b^\lambda] = \cap_{k=1}^\infty [a^{\lambda_k}, b^{\lambda_k}]$, then there exists a unique fuzzy number $V \in F(R^+)$ such that the family $[a^\lambda, b^\lambda]$ represents the λ-level sets of V.

Conversely, if $[a^\lambda, b^\lambda]$ are the the λ-level sets of a fuzzy number $V \in F(R^+)$, then there the conditions (i) and (ii) are satisfied.

In order to see that $\{(C)\int [\tilde{Y}]^\lambda d\mu \mid \lambda \in [0,1]\}$ define a fuzzy number in $F(R^+)$, we check (i) and (ii) as in the following:

(i) if $0 \le \lambda_1 \le \lambda_2 \le 1$, then we have $[\tilde{Y}]^{\lambda_1}(w) \subset [\tilde{Y}]^{\lambda_2}(w)$, for all $w \in \Omega$. That is, $y \in S([\tilde{Y}]^{\lambda_2})$ implies $y \in S([\tilde{Y}]^{\lambda_1})$. Thus, we have $(C)\int [\tilde{Y}]^{\lambda_2} d\mu \subset (C)\int [\tilde{Y}]^{\lambda_1} d\mu$, and

(ii) let $\{\lambda_n\}$ with $\lambda_n \uparrow \lambda$ which means a monotone increasing sequence, we have to see that

$$(C)\int [\tilde{Y}]^{\lambda} d\mu = \cap_{n=1}^\infty (C)\int [\tilde{Y}]^{\lambda_n} d\mu.$$

Therefore, we can obtain the following:

Theorem and Definition 4.3 Let $\tilde{Y}: \Omega \to F(R^+)$ be Choquet integrably bounded fuzzy number-valued random variable. Then there exists a uniquely fuzzy number $E_c(\tilde{Y})$ with λ-level sets $E_c([\tilde{Y}]^\lambda)$. $E_c(\tilde{Y})$ is called the Choquet expected value of $\tilde{Y}$.

5. The Choquet expected fuzzy number-valued utility function.

Expected utility theory combines linearity in probabilities and a utility function, which is either concave or convex if a dicision-maker is risk averse or seeking. However, maximization of expected utility as a criterion of choice among the alternatives involving risk fails to explain the existence of both insurance and lotteries. Given an utility function u, such that $u: \Omega \to R^+$, a fuzzy measure μ on $\mathfrak{I}$ and a set X of comonotonic prospects $x: S \to \Omega$, such that $x, x' \in X$ are comonotonic if and only if there are no $w_1, w_2 \in S$ such that $x(w_1) > x(w_2)$ and $x'(w_1) < x'(w_2)$, the Choquet integral permits the evaluation of the Choquet expected utility function as in the following theorem:

Definition 5.1 The preference relation $>$ on X is defined by

(1) $>$ is a complete and transitive, that is,

for $\forall x, y \in X$; $x > y$ *or* $y > x$ and

for $\forall x, y, z \in X$; $x > y$ and $y > z \rightarrow x > z$.

(2) $>$ is continuous, that is,

for $\forall x, y, z \in X$, $\forall \alpha \in (0,1)$,

$x > y$ and $y > z \rightarrow \exists \alpha, \beta \in (0,1)$ *such that* $\alpha x + (1-\alpha)z > y$ and $y > \beta x + (1-\beta)z$.

(3) $>$ is comonotonic independence, that is,

for $\forall x, y, z \in X$, $\forall \alpha \in (0,1)$,

$x > y \rightarrow \alpha x + (1-\alpha)z > \alpha y + (1-\alpha)z$.

Representation Theorem 5.2 ([28]) Let u be a utility function, then we have the following preference relation on X; for every $x, x' \in X$,

$$x > x' \leftrightarrow U(x) \geq U(x')$$

where U is defined as $U(x) = (C)\int_S u(x(s))\, d\mu(s)$ (Choquet integral with respect to μ).

Finally, we discuss a fuzzy number-valued utility function as in the following:

$$\tilde{u} : \Omega \rightarrow F(R^+)$$

We note that $[\tilde{u}]^\lambda$ is an interval number-valued utility function for all $\lambda \in [0,1]$. Then by Theorem 2.5, we have the Choquet expected fuzzy number-valued utility function $\tilde{U}$ as in the following:

$$[\tilde{U}]^\lambda(x) \equiv (C)\int_S [\tilde{u}]^\lambda(x(s))\, d\mu(s)$$

$$= [(C)\int_S [\tilde{u}]^\lambda_*(x(s))\, d\mu(s),\ (C)\int_S [\tilde{u}]^{\lambda *}(x(s))\, d\mu(s)]$$

and

$$\tilde{U}(x) = (C)\int_S \tilde{u}(x(s))\, d\mu(s).$$

Therefore, we can obtain the following theorem for fuzzy number-valued utility functions:

Representation Theorem 5.3([28]) Let $\tilde{u}$ be a fuzzy number-valued utility function, then we have the following preference relation on X; for every $x, x' \in X$,

$$x > x' \leftrightarrow \tilde{U}(x) \sqsupseteq \tilde{U}(x')$$

where $\tilde{U}$ is defined as $\tilde{U}(x) = (C)\int_S \tilde{u}(x(s))\, d\mu(s)$

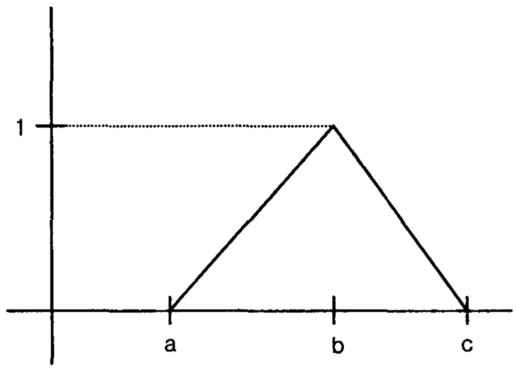
Remark 5.4 If S is a finite set $S = \{s_1, \cdots, s_n\}$, then we have the Choquet expected fuzzy number-valued utility function as in the following:

$$\tilde{U}(x) = \sum_{i=1}^{n}{}^{*} [\tilde{u}(x(s_{(i)})) - \tilde{u}(x(s_{(i-1)}))]\mu(A_{(i)})$$

$$= \sum_{i=1}^{n}{}^{*} \tilde{u}(x(s_{(i)}))[\mu(A_{(i)}) - \mu(A_{(i+1)})].$$

where $\sum^*$ is a fuzzy sum operation of fuzzy numbers and $A_{(i)} = \{(i), \cdots, (n)\}$ for all $i = 1, 2, \cdots n-1$ and $A_{(n+1)} = \varnothing$.

Example 5.5 Let a fuzzy number $A = (a, b, c)$ be defined by



Then we consider the following operations;
for two fuzzy numbers $A = (a, b, c)$, $B = (a', b', c')$,

$$A \sqsubseteq B \leftrightarrow a \leq a', b \leq b', c \leq c',$$
$$A \oplus B = (a + a', b + b', c + c'),$$
$$A \ominus B = (a - a', b - b', c - c'),$$
$$\lambda \odot A = (\lambda a, \lambda b, \lambda c).$$

And we consider a fuzzy measure μ defined by

$$\mu(A) = [\#(A)]^2$$

where $\#(A)$ is the number of elements of A. Now we consider a decision problem involving 4 cars, evaluated on 3 criteria as shown in Table 1: price, consumption and comfort.

Table 1 (s_1 ten million won, s_2 km/l)

	price(s_1)	consumption(s_2)	comfort(s_3)
Car1 (x)	1	10	Very Good
Car2 (y)	1	12	Good
Car3 (z)	3	8	Very Good
Car4 (w)	3	10	Good

In this case, suppose that we have the following fuzzy number-valued utility function ;

$$\tilde{u}(x(s_1)) = (0.8, 1, 1.2) = \tilde{u}(x(s_{(3)}))$$
$$\tilde{u}(x(s_2)) = (0.7, 0.9, 1.1) = \tilde{u}(x(s_{(2)}))$$
$$\tilde{u}(x(s_3)) = (0.5, 0.7, 0.9) = \tilde{u}(x(s_{(1)}))$$
$$\tilde{u}(y(s_1)) = (0.8, 1, 1.2) = \tilde{u}(y(s_{(1)}))$$
$$\tilde{u}(y(s_2)) = (1, 1.2, 1.4) = \tilde{u}(y(s_{(3)}))$$
$$\tilde{u}(y(s_3)) = (0.9, 1.1, 1.3) = \tilde{u}(y(s_{(2)}))$$
$$\tilde{u}(z(s_1)) = (0.9, 1.1, 1.3) = \tilde{u}(z(s_{(2)}))$$
$$\tilde{u}(z(s_2)) = (0.8, 1, 1.2) = \tilde{u}(z(s_{(1)}))$$
$$\tilde{u}(z(s_3)) = (1, 1.2, 1.4) = \tilde{u}(z(s_{(3)}))$$
$$\tilde{u}(w(s_1)) = (0.7, 0.9, 1.1) = \tilde{u}(w(s_{(2)}))$$
$$\tilde{u}(w(s_2)) = (0.8, 1, 1.2) = \tilde{u}(w(s_{(3)}))$$
$$\tilde{u}(w(s_3)) = (0.5, 0.7, 0.8) = \tilde{u}(w(s_{(1)})),$$

where $(\cdot)$ indicates a permutation on $\{1, 2, 3\}$ such that $s_{(1)} \leq s_{(2)} \leq s_{(3)}$.

Then we can calculate the Choquet expected fuzzy number-valued utility function as in the following:

$$\begin{aligned}\tilde{U}(x) &= \oplus_{i=1}^{3} \tilde{u}(x(s_{(i)}))[\mu(A_{(i)}) - \mu(A_{(i+1)})] \\ &= 5\odot(0.5, 0.7, 0.9)\oplus 3\odot(0.7, 0.9, 1.1) \\ &\quad \oplus 1\odot(0.8, 1, 1.2) \\ &= (5.4, 7.2, 9.0),\end{aligned}$$

and similarly we have

$$\tilde{U}(y) = (7.7, 9.5, 11.3),$$
$$\tilde{U}(z) = (7.7, 9.5, 11),$$
$$\tilde{U}(w) = (5.4, 7.2, 8.5).$$

Thus we can think that the consumer(decision maker) has the following preferences :

$\tilde{U}(x) \sqsubseteq \tilde{U}(y) \leftrightarrow x < y$ and

$\tilde{U}(w) \sqsubseteq \tilde{U}(z) \leftrightarrow w < z.$

That is, we can think that if price increases, so does the important of comfort and that it is more useful tool whenever some utility function is not clear. Furthermore, Example 5.5 is a decision problem for intertemporal preferences under an uncertain utility function.

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