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PRICING FLOATING-STRIKE LOOKBACK OPTIONS

Hangsuck Lee*

Abstract

A floating-strike lookback call option gives the holder the right to buy at the lowest price of the underlying asset. Similarly, a floating-strike lookback put option gives the holder the right to sell at the highest price. This paper will derive explicit pricing formulas for these floating-strike lookback options with flexible monitoring periods. The monitoring periods of these options start at an arbitrary date and end at another arbitrary date before maturity.

Key words: lookback option, floating strike, Brownian motion

1. Introduction

Lookback options are path-dependent contingent claims whose payoffs depend on the maximum (or minimum) of the underlying asset price over a certain period. A floating-strike lookback call option gives the holder the right to buy at the lowest price of the underlying asset. Similarly, a floating-strike lookback put option gives the holder the right to sell at the highest price. Goldman, Sosin and Gatto (1979) derived explicit pricing formulas for floating-strike lookback options where the highest (or lowest) price of the underlying asset is attained during the whole life of the options. Conze and Viswanathan (1991) derived explicit pricing formulas for partial floating-

* Department of Statistics and Actuarial Science, Soongsil University, Email:hslee@ssu.ac.kr

strike lookback options that give the holder the right to buy (or sell) at some percentage times the lowest (or highest) price.

Heynen and Kat (1994) suggest a way of reducing the price of these partial floating-strike lookback options while preserving some of their good qualities. However, those who have a specific view of the asset movement in a certain interval of the option life may be more interested in partial floating-strike lookback options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity. This paper will derive explicit pricing formulas for these generalized options with flexible monitoring periods.

2. Some basics

Let $S(t)$ denote the time-t price of an equity. Assume that the equity is constructed with all dividends reinvested. Assume that for $t \geq 0$,

$$S(t) = S(0)e^{X(t)}$$

where $\{X(t)\}$ is a Brownian motion with drift μ and diffusion coefficient σ and $X(0) = 0$. Thus the Brownian motion is a stochastic process with independent and stationary increments, and $X(t)$ has a normal distribution with mean μt and variance $\sigma^2 t$. Let $\boldsymbol{Z}$ = (Z_1, Z_2, Z_3) have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of the random vector $\boldsymbol{Z}$ is

$$\Phi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = \Pr(Z_1 \leq a, Z_2 \leq b, Z_3 \leq c).$$

3. Floating-Strike Lookback Call Option

Let us take a close look at the payoff of the floating-strike lookback call option. Assume that λ is some percentage and L is used for the maximum guaranteed strike price that can be interpreted as the highest asset price in a period of the past. The payoff of this option is

$$S(0)(e^{X(T)} - \lambda e^{\min(m(s,\,t),\,L)})_+, \tag{3.1}$$

where m(s, t) =min$\{X(\tau), s \le \tau \le t\}$. To simplify writing, we define all expectations in this and next sections as taken with respect to the risk-neutral measure. In other words, under this measure, the underlying stochastic process $\{X(\tau), \tau \ge 0\}$ is a Brownian motion with drift $r - \sigma^2/2$ and diffusion coefficient σ. By the fundamental theorem of asset pricing, the time-0 value of the payoff is

$$S(0)e^{-rT}\mathrm{E}[(e^{X(T)} - \lambda e^{\min(m(s,\,t),\,L)})_+]. \tag{3.2}$$

It can be shown that the time-0 value of the partial floating-strike lookback option call is

$$\begin{aligned} e^{-rT}\mathrm{E}[(e^{X(T)} - \lambda e^{\min(m(t),\,L)})_+] = -\{&-\Psi(f_1)\Psi(-e_1+\frac{\log\lambda}{\sigma\sqrt{T-t}}) \\ &+ \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_2(d_1+\frac{\log\lambda}{\sigma\sqrt{T}}, -e_1-\frac{\log\lambda}{\sigma\sqrt{T-t}}; -\sqrt{1-\frac{t}{T}}) \\ &- \lambda\frac{\sigma^2}{2r}e^{-rT}\; e^{2\frac{r}{\sigma^2}L}\Psi_2(f_1-\frac{2r\sqrt{t}}{\sigma}, d_1-\frac{2r\sqrt{T}}{\sigma}+\frac{\log\lambda}{\sigma\sqrt{T}}; \sqrt{\frac{t}{T}}) \\ &+ \lambda(1+\frac{\sigma^2}{2r})e^{-r(T-t)}\Psi(f_1)\Psi(-e_2+\frac{\log\lambda}{\sigma\sqrt{T-t}}) \\ &+ \lambda e^{-rT}e^{L}\Psi_2(-d_2+\frac{\log\lambda}{\sigma\sqrt{T}}, -f_2; \sqrt{\frac{t}{T}}) \\ &- \Psi_2(-d_1+\frac{\log\lambda}{\sigma\sqrt{T}}, -f_1; \sqrt{\frac{t}{T}})\}, \end{aligned} \tag{3.3}$$

where $\Psi(x) := \Phi(-x)$ and $\Psi_2(x, y; \rho) := \Phi_2(-x, -y; \rho)$ and where for i = 1 and 2, d_i denotes $\dfrac{-L+(r+(-1)^{i-1}\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, e_i is $\dfrac{(r+(-1)^{i-1}\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, and f_i is $\dfrac{-L+(r+(-1)^{i-1}\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$.

The time-0 price (3.2) can be expressed in the form of iterated expectations as follows:

$$e^{-rT}\mathrm{E}[e^{X(s)}\mathrm{E}[(e^{X(T)-X(s)} - \lambda e^{\min(m(s,t)-X(s),\, L-X(s))})_+ \,|\, X(s)]], \qquad (3.4)$$

which can be decomposed into the sum of two terms,

$$e^{-rs}\mathrm{E}[e^{X(s)}\mathrm{I}(X(s) < L)e^{-r(T-s)}\mathrm{E}[(e^{X(T)-X(s)} - \lambda e^{m(s,t)-X(s)})_+ \,|\, X(s)]]$$
$$+ e^{-rs}\mathrm{E}[e^{X(s)}\mathrm{I}(X(s) \geq L)e^{-r(T-s)}\mathrm{E}[(e^{X(T)-X(s)} - \lambda e^{\min(m(s,t)-X(s),\, L-X(s))})_+ \,|\, X(s)]]. \qquad (3.5)$$

First, let us consider the first term of (3.5). Applying the fact that the random vector $(m(s,t)-X(s), X(T)-X(s))$ is independent of $X(s)$ and has the same distribution as the random vector $(m(t-s), X(T-s))$, we see that the first term of (3.5) can be the product of two discounted expectations

$$e^{-rs}\mathrm{E}[e^{X(s)}\mathrm{I}(X(s) < L)]\; e^{-r(T-s)}\mathrm{E}[(e^{X(T-s)} - \lambda e^{m(t-s)})_+]. \qquad (3.6)$$

Applying the factorization formula derived by Gerber and Shiu (1996), we see that the first discounted expectation of (3.6) is

$$e^{-rs}\mathrm{E}[e^{X(s)}\mathrm{I}(X(s) < L)] = e^{-rs}\mathrm{E}[e^{X(s)}]\Pr(X(s) < L;\, 1)$$
$$= \Psi(g_1). \qquad (3.7)$$

From formula (3.5) with $L = 0$, $T = T-s$ and $t = t - s$, the second discounted expectation of (3.6) is

$$e^{-r(T-s)}\mathrm{E}[(e^{X(T-s)} - \lambda e^{m(t-s)})_+]$$
$$= -\{-\Psi(k_1)\Psi(-e_1+\frac{\log\lambda}{\sigma\sqrt{T-t}})$$
$$+ \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_2(h_1+\frac{\log\lambda}{\sigma\sqrt{T-s}}, -e_1-\frac{\log\lambda}{\sigma\sqrt{T-t}}; -\sqrt{\frac{T-t}{T-s}})$$
$$- \lambda\frac{\sigma^2}{2r}e^{-r(T-s)}\Psi_2(k_1 - \frac{2r\sqrt{t-s}}{\sigma}, h_1-\frac{2r\sqrt{T-s}}{\sigma}+\frac{\log\lambda}{\sigma\sqrt{T-s}}; \sqrt{\frac{t-s}{T-s}})$$
$$+ \lambda(1+\frac{\sigma^2}{2r})e^{-r(T-t)}\Psi(k_1)\Psi(-e_2+\frac{\log\lambda}{\sigma\sqrt{T-t}})$$

$$+\ \lambda e^{-r(T-s)}\Psi_2(-h_2+\frac{\log\lambda}{\sigma\sqrt{T-s}},-k_2;\ \sqrt{\frac{t-s}{T-s}})$$

$$-\ \Psi_2(-h_1+\frac{\log\lambda}{\sigma\sqrt{T-s}},-k_1;\ \sqrt{\frac{t-s}{T-s}})\}, \qquad (3.8)$$

where for $i = 1$ and 2, h_i is $\dfrac{(r+(-1)^{i-1}\frac{1}{2}\sigma^2)(T-s)}{\sigma\sqrt{T-s}}$, and k_i is $\dfrac{(r+(-1)^{i-1}\frac{1}{2}\sigma^2)(t-s)}{\sigma\sqrt{t-s}}$.

Let us consider the second term of (3.5). It follows from the partial lookback option formula (3.5) with $L = L - X(s)$, $T = T - s$ and $t = t - s$ that the second term of (3.5) becomes

$$-\{-\Psi_2(-g_1,f_1;-\sqrt{\frac{s}{t}})\Psi(-e_1+\frac{\log\lambda}{\sigma\sqrt{T-t}})$$

$$+\ \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_3(d_1+\frac{\log\lambda}{\sigma\sqrt{T}},-e_1-\frac{\log\lambda}{\sigma\sqrt{T-t}},-g_1;-\sqrt{1-\frac{t}{T}},-\sqrt{\frac{s}{T}},0)$$

$$-\ \lambda\frac{\sigma^2}{2r}e^{-rT}e^{2\frac{r}{\sigma^2}L}\Psi_3(f_1-\frac{2r\sqrt{t}}{\sigma},d_1-\frac{2r\sqrt{T}}{\sigma}+\frac{\log\lambda}{\sigma\sqrt{T}},-g_1+\frac{2r\sqrt{s}}{\sigma};\sqrt{\frac{t}{T}},-\sqrt{\frac{s}{t}},-\sqrt{\frac{s}{T}})$$

$$+\ \lambda(1+\frac{\sigma^2}{2r})e^{-r(T-t+s)}\Psi_2(-g_1,f_1;-\sqrt{\frac{s}{t}})\Psi(-e_2+\frac{\log\lambda}{\sigma\sqrt{T-t}})$$

$$+\ \lambda e^{-rT}e^{L}\Psi_3(-d_2+\frac{\log\lambda}{\sigma\sqrt{T}},-f_2,-g_2;\sqrt{\frac{t}{T}},\sqrt{\frac{s}{T}},\sqrt{\frac{s}{t}})$$

$$-\ \Psi_3(-d_1+\frac{\log\lambda}{\sigma\sqrt{T}},-f_1,-g_1;\sqrt{\frac{t}{T}},\sqrt{\frac{s}{T}},\sqrt{\frac{s}{t}})\} \qquad (3.9)$$

where $\Psi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = \Phi_3(-a, -b, -c; \rho_{12}, \rho_{13}, \rho_{23})$.

To calculate the time-0 value of the floating-strike lookback call option, the discounted expectation (3.2) is decomposed into the sum of the two discounted expectations of (3.5). The first discounted expectation is the product of (3.7) and (3.8), and the second discounted expectation is (3.9). Therefore, adding the two discounted expectations, we have the time-0 value of the floating-strike lookback call option with the monitoring period from time s to time t,

$$S(0)e^{-rT}\mathrm{E}[(e^{X(T)}-\lambda e^{\min(m(s,\,t),\,L)})_+]$$

$$= S(0)\{(3.9) + \Psi(g_1)(3.8)\}$$

$$=: V_{float}^{call}(S(0), \lambda, L, r, \sigma). \qquad (3.10)$$

Finally, let us discuss a duality relationship between the floating-strike call and put options. It can be shown that the put option formula is −1 times the call option formula (3.10) with its components Ψ, Ψ_2 and $\Psi_3$replaced by Φ, Φ_2 and Φ_3, respectively.

References

Conze, A. and Viswanathan (1991). "Path Dependent Options: the Case of Lookback Options," *Journal of Finance* 46:1893-1907.

Gerber, H.U. and Shiu, E.S.W. 1996. "Actuarial Bridges to Dynamic Hedging and Option Pricing," *Insurance: Mathematics and Economics* 18:183-218.

Goldman, M.B., Sosin, H.B. and Gatto, M.A. (1979). "Path Dependent Options: Buy at the Low, Sell at the High," *Journal of Finance* 34:1111-27.

Heynen, R.C. and Kat, H.M. (1994). "Partial Barrier Options," *The Journal of Financial Engineering* 3(4):253-274.

Heynen, R.C. and Kat, H.M. 1997. "Lookback Options - Pricing and Applications" in *Exotic Options: The State of the Art*, edited by Clewlow, C. and Strickland, C. London: International Thomson.

Lee, H. (2003). "Pricing Equity-Indexed Annuities with Path-Dependent Options," *Insurance: Mathematics and Economics* 33:677-690.