The Poisson Log-Bilinear Model of Forecasting Mortality and the Valuation of the Longevity Risk

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1 Introduction

The most widely used statistical methodology to forecast future mortality rates is the so-called Lee-Carter method (Lee and Carter, 1992), which assumes that the (theoretical) mortality rate $q_x(t)$ of an individual aged at x in year t can be expressed as

$$q_x(t) = \exp(\alpha_x + \beta_x x_t), \tag{1}$$

where the α_x and the β_x are age-specific parameters and x_t is a time-specific parameter. The standard approach to estimate the parameters is to model the observed mortality rate $m_x(t)$ as a log-bilinear regression

$$\log m_x(t) = \alpha_x + \beta_x x_t + \varepsilon_x(t), \quad t = 1, 2, \ldots, T; x = x_0, x_0 + 1, \ldots, \omega$$

with error terms $\varepsilon_x(t)$, and minimize the sum of squared errors

$$\sum_t \sum_x [\log m_x(t) - (\alpha_x + \beta_x x_t)]^2.$$

The above estimation procedure implicitly assumes that the errors $\varepsilon_x(t)$ follow the standard Normal distribution with an equal variance, which is quite unreal in that the mortality is much more variable at old ages than at younger ages. To incorporate the heteroskedasticity, Brouhns, Denuit and Vermunt (2002) and Renshaw and Haberman (2003) propose an alternative estimation procedure based on the Poisson log-bilinear

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modeling for the death count data.

In this talk we apply the Poisson methodology to Japanese mortality rates and evaluate the longevity risk in the living annuities by a bootstrapping procedure. The existent Poisson log-bilinear methodology may still need to be improved. In particular, we argue that the age-specific parameters α_x and the β_x may be better modeled as smooth functions of x, and thus we suggest a scheme based on the concept of the local likelihood (Loader, 1999) to impose smoothness on the age-specific parameters.

2 The Poisson log-bilinear model and its extension

The Poisson log-bilinear method models the death count D_{xt} recorded at age x and in year t as a realization from the Poisson distribution with mean $E_{xt}q_{xt}$:

$$D_{xt} \sim Poisson(E_{xt}q_{xt}), \quad t=1,2,\ldots,T;\ x=x_0,x_0+1,\ldots,\omega$$

where q_{xt} is given by (1) and the E_{xt} is the population aged x in year t. The parameters are estimated by maximizing the log-likelihood

$$\sum_{x,t}[D_{xt}(\alpha_x+\beta_x\kappa_t)-E_{xt}\exp\{\alpha_x+\beta_x\kappa_t\}]+constant.$$

The maximization can be done by iterating the following two steps

Step 1: Let $\{\kappa_t\}$ be given. Then, for each x, maximize

$$\sum_t[D_{xt}(\alpha_x+\beta_x\kappa_t)-E_{xt}\exp\{\alpha_x+\beta_x\kappa_t\}] \tag{2}$$

with respect to α_x and β_x

Step 2: Let $\{\alpha_x\}$ and $\{\beta_x\}$ be given. Then, for each t, maximize

$$\sum_x[D_{xt}(\alpha_x+\beta_x\kappa_t)-E_{xt}\exp\{\alpha_x+\beta_x\kappa_t\}]$$

with respect to κ_t

To put the smoothness on the age-specific parameters we propose an estimation scheme in which (2) is replaced with a smoothed version in the spirit of the local likelihood estimation of Loader (1999).

3 Evaluation of the longevity risk

As in the standard Lee-Carter method the estimated sequence $\{\widehat{x_t}, t=1,2,\ldots,T\}$ of the time-specific parameters is regarded as a time-series and fitted to an ARIMA model to project future values $\{\widetilde{x_t}, t=T+1, T+2,\cdots\}$ of the time specific parameter. The future mortalities are then predicted as

$$\widetilde{q_x}(t)=\exp(\widehat{a_x}+\widehat{\beta_x}\widetilde{x_t}), \quad t=T+1, T+2,\ldots,$$

where $\widehat{a_x}$ and $\widehat{\beta_x}$ are the estimated age-specific parameters. Figure 1 shows the $\{\widetilde{q_x}(t)\}$ for x =65 estimated and forecast from male mortality rates in Japan during 1970-2002. Given the projected mortality rates, the net single premium relating to a life annuity of an individual aged x at year t is then computed as

$$a_{xt} = \sum_{k\geq 0}\left\{\prod_{j=0}^{k}(1-\widetilde{q}_{x+j}(t+j))\right\}v^{k+1}$$

where v is a discount factor given by v = 1/(1 + r) with r being the interest rate.

As in Brouhns, Denuit and Vankeilegom (2004), starting from the observations (E_{xt}, D_{xt}), we create 100,000 Poisson bootstrap samples $\{E_{xt}, D_{xt}^{(n)}, n=1,2,\ldots,100{,}000\}$, where the $D_{xt}^{(n)}$'s are realizations from the Poisson distribution with mean

$$E_{xt}\widehat{q_x}(t) \quad \text{with} \quad \widehat{q_x}(t)=\exp(\widehat{a_x}+\widehat{\beta_x}\widehat{x_t}), \quad t=1,2,\ldots,T.$$

Figure 2 shows the histogram of the 100,000 such realizations for x =65, from which we can evaluate the longevity risk in the living annuity by calculating the risk measures such as the value-at-risk and the expected short fall.

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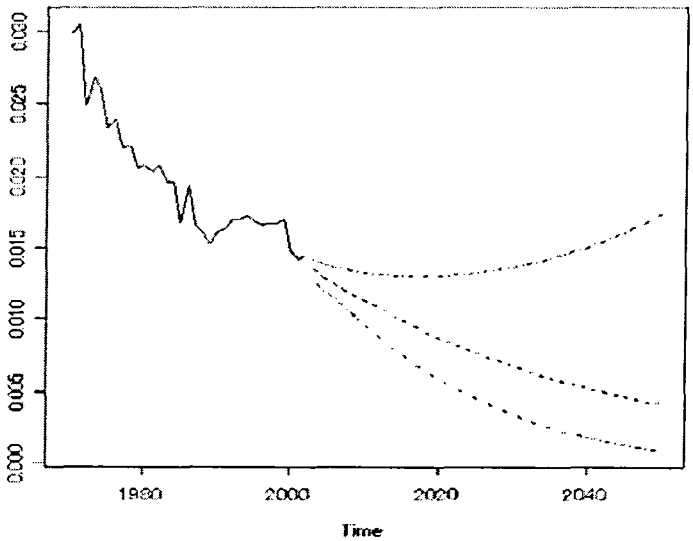

Figure 1: The forecast of the time-specific parameters κ_t

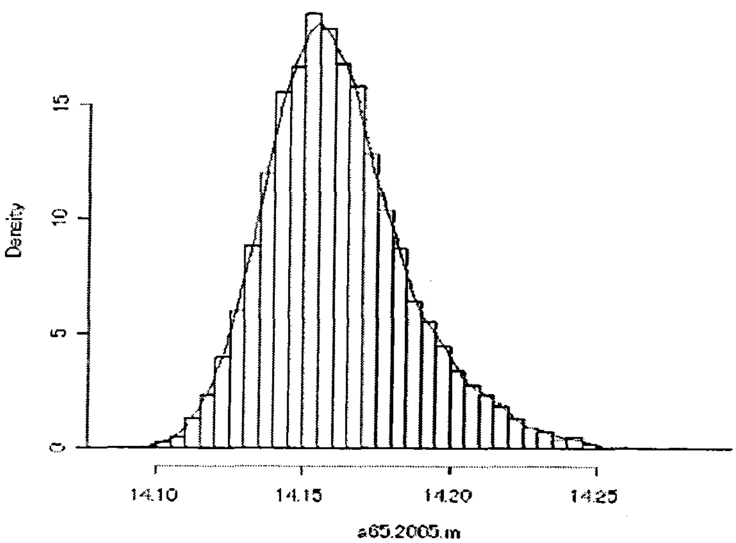

Figure 2: The histogram of 100, 000 realizations of a_{xt}'s