# Monitoring with VSR Charts and Change Point Estimation

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**Abstract**

Knowing the time of the process change could lead to quicker identification of the responsible special cause and less process down time, and it could help to reduce the probability of incorrectly identifying the special cause. In this paper, we propose a MLE of the process change point when control charts with the fixed sampling rate (FSR) scheme or the variable sampling rate (VSR) scheme monitor a process to detect changes in the process mean and/or variance of a normal quality variable.

## 1. Introduction

When a control chart signals that a special cause is present, process engineers must initiate a search for and an identification of the special cause. Knowing the time of the process change could lead to identify the special cause more quickly, and to take the appropriate actions immediately to improve quality. Consequently, estimating the time of the process change would be useful to process engineers.

The CUSUM and the EWMA charts provide built-in change point estimators from the behavior of the past plots on the control chart. Samuel, Pignatiello, and Calvin (1998) considered the use of a MLE of the change point for a step change in a normal process mean, and investigated its performance when used after a signal from a $\bar{X}$ chart. Pignatiello and Samuel (2001) considered using the MLE of the change point instead of the built-in change point estimator when either the CUSUM or the EWMA chart issued a signal. They concluded that the performance of the MLE appears to be better than the built-in estimators over the range of magnitudes of the change considered.

The traditional approach to sampling for a control chart is to use a FSR, however in recent years there have been investigations of control charts with a VSR. Approaches to vary the sampling rate are a variable sampling interval (VSI) chart and a variable sample size (VSS) chart. General VSR charts allow both the sample size and the sampling interval as a function of the sample results from the process.

In this paper, we propose MLEs of the process change point when FSR charts monitor a process to detect changes in the process mean and/or variance of a normal quality variable, and generalized the proposed estimators of FSR charts to use in VSR charts. By the extensive simulation we investigate the performance of these estimators.

## 2. Description of the VSR chart

Consider the problem of monitoring a process and let $X$ represent the process quality variable being measured with mean $\mu_0$ and variance $\sigma_0^2$. Let $N_t$ be the sample size used at the $t$-th sampling time and $H_t$ be the sampling interval used between sampling times $k-1$ and $t$. Let $Z_t = \sqrt{N_t}\,(\overline{X}_t - \mu_0)/\sigma_0$, the standardized sample mean, and let $Y_t$ be the control statistic computed for sample $t$. We assume that $Y_t$ can be expressed as a function

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of $Z_t$. The monitoring procedure is to signal at time $t$ if $|Y_t| \geq c$ for a control limit $c$. The values of $N_t$ and $H_t$ for the sample $t$ are determined according to the value of the previous statistic, $Y_{t-1}$.

In this paper we consider only two sampling intervals, $h_1$ and $h_2$ ($h_2 < h_1$), for the VSI scheme. Then for $t \geq 2$, the sampling interval $H_t$ can be represented as

$$H_t = \begin{cases} h_1 & \text{if } |Y_{t-1}| < th \\ h_2 & \text{if } th \leq |Y_{t-1}| < c \end{cases},$$

where $th$ denotes the threshold limit to switch between the two sampling intervals.

We also consider only two possible sample sizes, $n_1$ and $n_2$ ($n_1 < n_2$), for simplicity. Then for $t \geq 2$, the sample size $N_t$ can be represented as

$$N_t = \begin{cases} n_1 & \text{if } |Y_{t-1}| < tn \\ n_2 & \text{if } tn \leq |Y_{t-1}| < c \end{cases},$$

where $tn$ denotes the threshold limit to switch between the two sample sizes.

When $th \neq tn$, the sampling interval $H_t$ and the sample size $N_t$ can be represented together as

$$(H_t, N_t) = \begin{cases} (h_1, n_1) & \text{if } |Y_{t-1}| < \min\{th, tn\} \\ (h_*, n_*) & \text{if } \min\{th, tn\} \leq |Y_{t-1}| < \max\{th, tn\} \\ (h_2, n_2) & \text{if } \max\{th, tn\} \leq |Y_{t-1}| < c \end{cases},$$

where $(h_*, n_*) = (h_1, n_2)$ if $\min\{th, tn\} = tn$ and $(h_*, n_*) = (h_2, n_1)$ if $\min\{th, tn\} = th$.

The performance of a VSR control chart can be evaluated by considering the number of samples, the number of individual observations, and the time required by the chart signal. Define the average number of samples to signal (ANSS) to be expected number of samples taken from a specified starting time to the time that the chart signals. Similarly, define the average number of observations to signal (ANOS) to be expected number of observations taken from a specified starting time to the time that the chart signals. Also define the average time to signal (ATS) to be the expected time from a specified starting time to the time that the chart signals. Then the average sampling interval is defined to be $\bar{h}$ = ATS/ANSS, and the average sample size is defined to be $\bar{n}$ = ANOS/ANSS.

### 3. Control chart for monitoring the process mean and variance

Assume that $X$ follows a normal distribution with mean $\mu$ and variance $\sigma^2$, and assume that this process starts with the in-control value for the mean $\mu_0$, and with the in-control value for the standard deviation $\sigma_0$. The objective process monitoring is to detect any special cause that changes $\mu$ from $\mu_0$, and/or changes $\sigma$ from $\sigma_0$. In this paper we only consider the problem of detecting an increase in $\sigma$.

The Shewhart $\bar{X}$ chart for detecting changes in $\mu$ is based on the control statistic $Y_t = Z_t$. The EWMA chart for detecting changes in $\mu$ is based on the control statistic $Y_t = \lambda Z_t + (1-\lambda)Y_{t-1}$, for a weight $\lambda (0 < \lambda \leq 1)$ and $Y_0 = 0$. Both the $\bar{X}$ chart and the EWMA chart signal if $|Y_t| \geq c_m$, where $c_m$ denotes the control limit for monitoring the mean.

The decision rules at sampling time $t$ for these charts with the VSR scheme can be defined as follow. Use the long sampling interval $h_1$ if $|Y_{t-1}| < th_m$ and the short sampling interval $h_2$ if $th_m \leq |Y_{t-1}| < c_m$, where $th_m$ denotes the threshold limit to switch between the two sampling intervals for monitoring the mean. Use the small sample size $n_1$ if $|Y_{t-1}| < tn_m$ and the large sample size $n_2$ if $tn_m \le |Y_{t-1}| < c_m$, where $tn_m$ denotes the threshold limit to switch between the two sample sizes for monitoring the mean.

To use of EWMA-based statistics for detecting an increase in $\sigma$, we consider an EWMA chart of the squared deviations from target, which is usually called an exponentially weighted moving variance (EWMV) chart (see, e.g., MacGregor and Harris (1993)). The control statistic for this chart is defined as $Y_t = \lambda Z_t^2 + (1-\lambda)\max\{Y_{t-1}, 1\}$, where $Y_0 = 1$. A signal is given if $Y_t \ge c_s$, where $c_s$ denotes the control limit for monitoring the standard deviation.

The decision rules at sampling time $t$ for an EWMV chart with VSR scheme can be defined as follow. Use the long sampling interval $h_1$ if $Y_{t-1} < th_s$ and the short sampling interval $h_2$ if $th_s \le Y_{t-1} < c_s$, where $th_s$ denotes the threshold limit to switch between the two sampling intervals for monitoring the standard deviation. Use the small sample size $n_1$ if $Y_{t-1} < tn_s$ and the large sample size $n_2$ if $tn_s \le Y_{t-1} < c_s$, where $tn_s$ denotes the threshold limit to switch between the two sample sizes for monitoring the standard deviation.

## 4. Estimation of the process change point

We assume that process parameter changes are occurred after an unknown point in time $\tau$ (known as the process change point), and let $T$ denote the number of samples from the start of monitoring to the time that the chart signals. Then $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_\tau$ are the subgroup averages that are taken from the in-control process, whereas $\bar{X}_{\tau+1}, \bar{X}_{\tau+2}, \cdots, \bar{X}_T$ are from the changed process.

Samuel, Pignatiello, and Calvin (1998) proposed a MLE of the change point of a normal process mean, and investigated its performance when a FSR $\bar{X}$ chart issues a signal. Their proposed estimators of $\tau$, $\hat{\tau}_{\mu-FSR}$, would be

$$\hat{\tau}_{\mu-FSR} = \max_{0\le t<T}\{t : (T-t)\,(\bar{Z}_{t+1,T})^2\}, \tag{1}$$

where $\bar{Z}_{t+1,T} = \sum_{i=t+1}^{T} Z_i/(T-t)$ is the overall average of the standardized sample mean for the last $T-t$ subgroups.

However, this estimator in equation (1) is not a MLE when the sample sizes are not equal. We proposed a MLE of the change point of a normal process mean, $\hat{\tau}_{\mu-VSR}$, which can be applied in cases where the sample sizes are variable, as follows:

$$\hat{\tau}_{\mu-VSR} = \max_{0\le t<T}\{t : N_{t+1,T}\,(\bar{Z}^*_{t+1,T})^2\}, \tag{2}$$

where $N_{t+1,T} = \sum_{i=t+1}^{T} N_i$ and $\bar{Z}^*_{t+1,T} = \sum_{i=t+1}^{T} \sqrt{N_i} Z_i/N_{t+1,T}$. Note that if $N_i = \bar{n}$ for all $i$, the MLE in equation (2) reduces to the MLE in equation (1).

We proposed a MLE of the change point of a normal process standard deviation, $\hat{\tau}_\sigma$, which can be applied when a signal is given on the FSR chart for monitoring $\sigma$, as follows:

$$\hat{\tau}_\sigma = \min_{0\le t<T}\{t : (T-t)(\ln S^2_{t+1,T} + 1) + \sum_{i=1}^{t} Z_i^2\}, \tag{3}$$

where $S^2_{t+1,T} = \sum_{i=t+1}^{T} Z_i^2/(T-t)$. Note that this estimator in equation (3) is also a MLE when the sample sizes are not equal. Therefore this MLE can be applied when a VSR chart as well as a FSR chart for monitoring $\sigma$ issues a signal.

## 5. Performance of the estimator in monitoring $\mu$ and $\sigma$

Consider the situation in which the quality variable $X$ follows a normal distribution and the objective is to detect shifts in $\mu$ and/or $\sigma$. For this purpose, the $\bar{X}$ chart, the EWMA chart, and the combined EWMA and EWMV chart will be considered.

For the comparisons, the charts are set up so that the in-control ANSS is 370.4. This choice for the ANSS implies that ATS/$\bar{h}$ = ANSS/$\bar{n}$ = 370.4 when in-control. Both the EWMA chart and the EWMV chart use $\lambda = 0.1$. For each chart the values of the control limits and the threshold limits are adjusted so that the required in-control properties are obtained. These values are obtained through trial and error.

The VSR chart is considered with $(h_1/\bar{h}, h_2/\bar{h}) = (1.9, 0.1)$ and $(n_1/\bar{n}, n_2/\bar{n}) = (0.6, 4.0)$. Reynolds and Arnold (2001) have shown that these ratios provide good statistical performance for VSI and VSS charts. For convenience we assume that the shift in $\mu$ is measured in units of $\delta = \sqrt{\bar{n}}(\mu - \mu_0)/\sigma_0$, and the shift in $\sigma$ is expressed in terms of $\gamma = \sigma/\sigma_0$ $(\gamma \geq 1)$.

Now we use Monte Carlo simulation to study the performance of the change point estimators. The process change points are generated by geometric distribution with mean $E(\tau) = 100$. The results from the 100,000 simulation runs for various sizes of change in $\mu$ and/or $\sigma$ are Table 1 to Table 2. Table 1 is for the case of FSR charts, and Table 2 is for the case of VSR charts. The column labeled $\bar{d}$ denotes the average value of $d_i = \tau_i - \hat{\tau}_i$, which is the average difference between the true change point value and the change point estimator. The change point estimators can be obtained by equation (1) or (2) for the case when control charts detecting changes in $\mu$. We use equation (1) when the sample sizes are equal, and use equation (2) when the sample sizes are variable. The change point estimators can be obtained also by equation (3) for the case when control charts detecting changes in $\sigma$ give a signal. The column labeled $s_d$ denotes the standard error of $\bar{d}$.

A comparison of $ATS/\bar{n}$ of Table 1 and Table 2 shows that combining the EWMA chart and the EWMV chart give more better performance than using the $\bar{X}$ chart or the EWMA chart separately. Especially when we use the EWMA chart alone for monitoring $\mu$ but there are changes only in $\sigma$, that is the cases for $\delta = 0$ and $\gamma > 1$, these charts do not perform well. Also in these cases the MLE based on equation (1) appears to be much biased in estimating the process change point. This is because the MLE based on equation (1) is the change point estimator of $\mu$.

The results in Table 1 and Table 2 show that the proposed MLE provides good performance when it is used with any sampling rate scheme except the cases for small changes in both $\mu$ and $\sigma$, and for changes only in $\sigma$. As previously stated, the combined EWMA and EWMV chart is recommended to prepare for the process changes in $\sigma$. Using the combination of two charts can substantially reduce the time required to detect changes in $\mu$ and/or $\sigma$, and yields better results in estimating the time of the process change. We note that as the amount of $\delta$ and/or $\gamma$ increases, the bias of the proposed MLE decreases greatly.

## 6. Conclusions

When a control chart signals that a special cause is present, the signal does not provide process engineers with what caused the process to change or when the process change actually occurred. Knowing the time of the process change would help process engineers in their search for the special cause. Consequently, the estimation for the process change point can fill an important role in obtaining information for finding special causes.

In this paper, we have proposed several MLEs of the process change point when control charts with the FSR scheme and the VSR scheme signal a change in $\mu$ and/or $\sigma$ of a normal quality variable. We also have discussed the performance of the proposed MLEs when it is used with various control charts and various sampling rate schemes. The results from the extensive simulation that the proposed MLEs provide good performance over the range of shifts.

| | | $\bar{X}$ chart | | | EWMA chart | | | EWMA chart EWMV chart | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $\gamma$ | $ATS/h$ | $d$ | $s_d$ | $ATS/h$ | $d$ | $s_d$ | $ATS/h$ | $d$ | $s_d$ |
| 0.00 | 1.00 | 370.53 | | | 370.74 | | | 370.75 | | |
| 0.00 | 1.25 | 60.98 | -57.53 | 0.1926 | 98.51 | -88.27 | 0.3096 | 44.42 | -24.34 | 0.1291 |
| 0.00 | 1.50 | 22.02 | -19.33 | 0.0716 | 46.53 | -38.97 | 0.1463 | 16.20 | -4.54 | 0.0626 |
| 0.00 | 2.00 | 7.48 | -5.46 | 0.0315 | 19.34 | -14.52 | 0.0610 | 6.48 | -0.13 | 0.0403 |
| 0.00 | 3.00 | 3.17 | -1.64 | 0.0182 | 7.97 | -5.02 | 0.0265 | 3.14 | 0.42 | 0.0283 |
| 0.50 | 1.00 | 154.99 | -5.15 | 0.0717 | 27.29 | -5.45 | 0.0752 | 32.51 | -6.20 | 0.0754 |
| 0.50 | 1.25 | 39.59 | -13.08 | 0.0747 | 23.12 | -10.74 | 0.0710 | 20.32 | -7.29 | 0.0670 |
| 0.50 | 1.50 | 17.32 | -10.68 | 0.0521 | 19.75 | -11.79 | 0.0617 | 12.16 | -2.74 | 0.0531 |
| 0.50 | 2.00 | 6.81 | -4.57 | 0.0265 | 14.03 | -9.25 | 0.0449 | 5.98 | -0.02 | 0.0384 |
| 0.50 | 3.00 | 3.09 | -1.55 | 0.0174 | 7.49 | -4.58 | 0.0248 | 3.07 | 0.44 | 0.0282 |
| 1.00 | 1.00 | 43.97 | -0.51 | 0.0237 | 9.48 | 1.17 | 0.0421 | 10.19 | 1.27 | 0.0411 |
| 1.00 | 1.25 | 18.09 | -2.39 | 0.0285 | 9.36 | -0.88 | 0.0384 | 8.96 | 0.04 | 0.0446 |
| 1.00 | 1.50 | 10.48 | -3.41 | 0.0291 | 9.15 | -2.58 | 0.0351 | 7.37 | -0.25 | 0.0414 |
| 1.00 | 2.00 | 5.49 | -2.91 | 0.0214 | 8.48 | -4.07 | 0.0284 | 4.92 | 0.21 | 0.0356 |
| 1.00 | 3.00 | 2.92 | -1.39 | 0.0152 | 6.30 | -3.47 | 0.0215 | 2.93 | 0.45 | 0.0278 |
| 2.00 | 1.00 | 6.30 | 0.24 | 0.0146 | 4.12 | 0.99 | 0.0233 | 3.68 | 2.11 | 0.0331 |
| 2.00 | 1.25 | 4.70 | -0.04 | 0.0168 | 4.18 | 0.59 | 0.0223 | 3.53 | 1.63 | 0.0320 |
| 2.00 | 1.50 | 3.98 | -0.37 | 0.0188 | 4.23 | 0.13 | 0.0223 | 3.39 | 1.25 | 0.0315 |
| 2.00 | 2.00 | 3.17 | -0.88 | 0.0150 | 4.29 | -0.78 | 0.0191 | 3.05 | 0.70 | 0.0289 |
| 2.00 | 3.00 | 2.40 | -0.89 | 0.0126 | 4.18 | -1.62 | 0.0162 | 2.45 | 0.40 | 0.0255 |
| 3.00 | 1.00 | 2.00 | 0.39 | 0.0171 | 2.74 | 0.52 | 0.0147 | 1.96 | 1.76 | 0.0287 |
| 3.00 | 1.25 | 2.00 | 0.23 | 0.0140 | 2.78 | 0.40 | 0.0169 | 1.98 | 1.51 | 0.0275 |
| 3.00 | 1.50 | 2.00 | 0.08 | 0.0143 | 2.81 | 0.22 | 0.0131 | 2.01 | 1.32 | 0.0280 |
| 3.00 | 2.00 | 1.99 | -0.21 | 0.0127 | 2.89 | -0.13 | 0.0137 | 2.04 | 0.82 | 0.0237 |
| 3.00 | 3.00 | 1.91 | -0.47 | 0.0151 | 3.00 | -0.75 | 0.0143 | 1.98 | 0.38 | 0.0200 |
| $c_m$ | | | 3.000 | | | 0.620 | | | 0.671 | |
| $c_s$ | | | | | | | | | 2.261 | |

Table 1. ATS/$\bar{h}$, the average difference between $\tau$ and $\hat{\tau}$, and the associated standard errors in the $\bar{X}$ chart, the EWMA chart, the combined EWMA and EWMV chart with the FSR scheme

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| | | $\bar{X}$ chart | | | EWMA chart | | | EWMA chart EWMV chart | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | $\gamma$ | $ATS/h$ | $d$ | $s_d$ | $ATS/h$ | $d$ | $s_d$ | $ATS/h$ | $d$ | $s_d$ |
| 0.00 | 1.00 | 370.60 | | | 370.51 | | | 370.27 | | |
| 0.00 | 1.25 | 50.76 | -57.31 | 0.1929 | 86.51 | -88.90 | 0.3131 | 25.50 | -24.33 | 0.1288 |
| 0.00 | 1.50 | 15.80 | -19.20 | 0.0710 | 37.40 | -39.10 | 0.1462 | 7.01 | -4.58 | 0.0610 |
| 0.00 | 2.00 | 4.27 | -5.50 | 0.0280 | 13.91 | -14.62 | 0.0619 | 2.48 | -0.07 | 0.0421 |
| 0.00 | 3.00 | 1.31 | -1.63 | 0.0160 | 4.86 | -5.07 | 0.0259 | 1.12 | 0.46 | 0.0287 |
| 0.50 | 1.00 | 115.01 | -10.30 | 0.0855 | 13.84 | -7.69 | 0.0618 | 16.77 | -10.57 | 0.0683 |
| 0.50 | 1.25 | 25.94 | -13.24 | 0.0688 | 13.39 | -9.33 | 0.0572 | 9.69 | -7.95 | 0.0599 |
| 0.50 | 1.50 | 10.46 | -9.31 | 0.0475 | 12.26 | -9.85 | 0.0529 | 5.24 | -3.09 | 0.0505 |
| 0.50 | 2.00 | 3.59 | -4.19 | 0.0250 | 8.99 | -8.34 | 0.0418 | 2.33 | 0.02 | 0.0399 |
| 0.50 | 3.00 | 1.25 | -1.50 | 0.0169 | 4.41 | -4.45 | 0.0239 | 1.11 | 0.45 | 0.0299 |
| 1.00 | 1.00 | 10.48 | -2.95 | 0.0255 | 4.93 | -1.77 | 0.0346 | 5.09 | -2.08 | 0.0366 |
| 1.00 | 1.25 | 6.21 | -3.13 | 0.0242 | 5.21 | -2.56 | 0.0310 | 4.25 | -2.08 | 0.0363 |
| 1.00 | 1.50 | 4.15 | -2.98 | 0.0228 | 5.28 | -3.19 | 0.0264 | 3.24 | -1.21 | 0.0376 |
| 1.00 | 2.00 | 2.29 | -2.26 | 0.0206 | 4.98 | -3.76 | 0.0254 | 1.96 | 0.02 | 0.0346 |
| 1.00 | 3.00 | 1.09 | -1.23 | 0.0160 | 3.46 | -3.23 | 0.0202 | 1.04 | 0.41 | 0.0281 |
| 2.00 | 1.00 | 0.84 | -0.51 | 0.0136 | 1.81 | -0.78 | 0.0130 | 1.36 | 0.07 | 0.0275 |
| 2.00 | 1.25 | 0.93 | -0.60 | 0.0141 | 1.93 | -0.99 | 0.0140 | 1.43 | 0.01 | 0.0276 |
| 2.00 | 1.50 | 0.96 | -0.68 | 0.0152 | 2.07 | -1.18 | 0.0146 | 1.44 | 0.08 | 0.0304 |
| 2.00 | 2.00 | 0.93 | -0.77 | 0.0165 | 2.22 | -1.46 | 0.0126 | 1.27 | 0.21 | 0.0299 |
| 2.00 | 3.00 | 0.73 | -0.75 | 0.0127 | 2.04 | -1.68 | 0.0122 | 0.89 | 0.25 | 0.0234 |
| 3.00 | 1.00 | 0.28 | -0.35 | 0.0131 | 0.94 | -0.72 | 0.0048 | 0.56 | 0.66 | 0.0284 |
| 3.00 | 1.25 | 0.37 | -0.37 | 0.0133 | 1.01 | -0.79 | 0.0051 | 0.66 | 0.48 | 0.0248 |
| 3.00 | 1.50 | 0.44 | -0.42 | 0.0147 | 1.08 | -0.83 | 0.0076 | 0.72 | 0.39 | 0.0246 |
| 3.00 | 2.00 | 0.51 | -0.46 | 0.0125 | 1.22 | -0.93 | 0.0071 | 0.79 | 0.28 | 0.0251 |
| 3.00 | 3.00 | 0.51 | -0.49 | 0.0122 | 1.30 | -1.07 | 0.0087 | 0.73 | 0.18 | 0.0216 |
| $c_m$ | | 3.000 | | | 0.620 | | | 0.671 | | |
| $c_s$ | | | | | | | | 2.261 | | |
| $th_m$ / $tn_m$ | | 0.672 / 1.555 | | | 0.149 / 0.343 | | | 0.219 / 0.403 | | |
| $th_s$ / $tn_s$ | | | | | | | | 1.213 / 1.591 | | |

Table 2. ATS/$\bar{h}$, the average difference between $\tau$ and $\hat{\tau}$, and the associated standard errors in the $\bar{X}$ chart, the EWMA chart, the combined EWMA and EWMV chart with the VSR scheme