Review on the Limiting Behavior of Tail Series of Independent Summands

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ABSTRACT

For the almost certainly convergent series S_n of independent random variables, the limiting behavior of tail series $T_n \equiv S - S_{n-1}$ is reviewed. More specifically, tail series strong laws of large number and tail series weak laws of large numbers will be introduced, and their relationship will be investigated. Then, the relationship will also be extended to the case of Banach space valued random elements, by investigating the duality between the limiting behavior of the tail series of random variables and that of random elements.

Keywords: Real separable Banach space, Rate of convergence, Almost certain convergence, Convergence in probability, Tail series

1. Introduction

The primary objective of the current work is to review the previous studies on the *rate* of convergence for an almost sure convergent series. Let

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$\{X_n, n \geq 1\}$ be random variables and let $\{V_n, n \geq 1\}$ be Banach space valued random elements defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and, as usual, their partial sums are denoted by $S_n = \sum_{i=1}^{n} X_i$ (resp., $S_n = \sum_{i=1}^{n} V_i$), $n \geq 1$. If the series S_n converges a.c. to a random variable (resp., a Banach space valued random element) S, define (set $S_0 = 0$) $T_n = S - S_{n-1}, n \geq 1$. Then the *tail series* $\{T_n, n \geq 1\}$ is a well-defined sequence with

$$T_n \to 0 \text{ a.c.} \tag{1.1}$$

The sequence $\{X_n, n \geq 1\}$ (resp., $\{V_n, n \geq 1\}$) is said to obey the tail series *weak law of large numbers* (WLLN) with positive norming constants $\{b_n, n \geq 1\}$ if the tail series $\{T_n, n \geq 1\}$ is well defined and for a given sequence of positive constants with $b_n = o(1)$,

$$\frac{T_n}{b_n} \xrightarrow{P} 0, \tag{1.2}$$

and is also said to obey the tail series *strong law of large numbers* (SLLN) with norming constants $\{b_n, n \geq 1\}$ if the tail series T_n is well defined and for a given sequence of positive constants with $b_n = o(1)$,

$$\frac{T_n}{b_n} \to 0 \text{ a.c.} \tag{1.3}$$

In this paper, for independent summands of random variables (resp., Banach spaced valued random elements), we shall be concerned with the rate in which S_n converges to S, or equivalently, in which the tail series T_n converges to 0.

2. Tail series of independent random variables

Pioneering work on the limiting behavior of the tail series of random variables $\{T_n, n \geq 1\}$ was conducted by Chow and Teicher (1973) wherein they obtained a tail series law of the iterated logarithm (LIL). After numerous other investigations on the tail series LIL problem had been made, the tail series SLLN problem was studied by Klesov (1983, 1984), Mikosch (1990), Nam and Rosalsky (1995a) and Nam (2004).

Recalling that (1.1) is equivalent to

$$\sup_{j \geq n} |T_j| \xrightarrow{P} 0,$$

Nam and Rosalsky (1995b) provided various sets of conditions in order for the limit law

$$\frac{\sup_{j \geq n} |T_j|}{b_n} \xrightarrow{P} 0 \tag{2.1}$$

to hold for a given sequence of positive constants $\{b_n, n \geq 1\}$, and then this result was generalized by Sung and Volodin (2001). When $0 < b_n \downarrow$, Nam and Rosalsky (1995b) observed it follows from

$$\frac{\sup_{j \geq n} |T_j|}{b_n} \leq \sup_{j \geq n} \frac{|T_j|}{b_j}$$

that the tail series SLLN (1.3) implies the limit law (2.1) and that (1.3) is indeed equivalent to the apparently stronger limit law

$$\frac{\sup_{j \geq n} |T_j|}{b_n} \to 0 \text{ a.c.} \tag{2.2}$$

They also provided an example wherein the limit law (2.1) holds with $0 < b_n \downarrow$ but SLLN (1.3) fails. In their follow-up article, Nam and Rosalsky (1996) proved apropos of the sequence of random variables that the tail series WLLN (1.2) and the limit law (2.1) are indeed equivalent when $0 < b_n \downarrow$, thereby establishing the validity of a conjecture posed by Nam and Rosalsky (1995b). Moreover, Nam and Rosalsky (1996) provided an example showing that without the monotonicity condition on $\{b_n, n \geq 1\}$, the tail series SLLN (1.3) (as well as tail series WLLN (1.2)) does not imply either of the limit laws (2.1) or (2.2).

3. Tail series of independent random elements

Throughout this section, $\{V_n, n \geq 1\}$ is a sequence of independent random elements taking values in a real separable Banach space $\mathcal{X}$ with norm $||\cdot||$. The limiting behavior of the tail series of Banach space valued random elements was investigated by Dianliang (188, 1991). Rosalsky and Rosenblatt (1997) provided conditions in their theorem in order for the limit law

$$\frac{\sup_{j \geq n} ||T_j||}{b_n} \xrightarrow{P} 0 \tag{3.1}$$

to hold for a given sequence of positive constants $\{b_n, n \geq 1\}$, thereby extending Theorem 4 of Nam and Rosalsky (1995b) which pertained to only the random variable case. When $0 < b_n \downarrow$, Rosalsky and Rosenblatt (1997) observed that the tail series SLLN (1.3) implies the limit law (3.1) and that (1.3) is equivalent to the apparently stronger limit law

$$\frac{\sup_{j \geq n} ||T_j||}{b_n} \to 0 \text{ a.c.}$$

Nam et al. (1999) extended the Nam and Rosalsky (1996) equivalence between the tail series WLLN (1.2) and the limit law (3.1) from the random variable case to the case of Banach space valued random elements.

References

(1) Chow, Y.S. and Teicher, H., Iterated logarithm laws for weighted averages, *Z. Wahrsch. Verw. Gebiete,* **26** (1973), 87–94.

(2) Chow, Y.S. and Teicher, H., *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed., Springer, New York, 1997.

(3) Dianliang, D., The law of iterated logarithm of tail sums of Banach space valued random variables, *Acta Sci. Natur. Jilin Univ.,* **2** (1988), 15–24, In Chinese.

(4) Dianliang, D., The estimation of the superior limit in the law of the iterated logarithm for tail sums, *Northeastern Math. J.* (People's Republic of China), **7** (1991), 265–274.

(5) Klesov, O.I., Rate of convergence of series of random variables, *Ukrain. Mat. Zh.,* **35** (1983), 309–314 (In Russian; English translation in *Ukrainian Math. J.,* **35** (1983), 264–268).

(6) Klesov, O.I., Rate of convergence of some random series, *Teor. Veroyatnost. i Mat. Statist.,* **30** (1984), 81–92 (In Russian; English translation in *Theor. Probab. Math. Statist.,* **30** (1985), 91–101).

(7) Mikosch, T., Almost sure behavior of tail series in functional spaces, *Analysis Math.,* **16** (1990), 123–133.

(8) Nam, E., On the order of growth of convergent series of independent random variables, *J. Appl. Math. Stochastic Anal.,* **2004:2** (2004), 159–168.

(9) Nam, E. and Rosalsky, A., On the almost certain rate of convergence of series of independent random variables, *J. Korean Statist. Soc.,* **24** (1995), 91–109.

(10) Nam, E. and Rosalsky, A., On the rate of convergence of series of random variables, *Teor. Īmovīrnost. ta Mat. Statist.,* **52** (1995), 120–131 (In Ukrainian; English translation in *Theor. Probab. Math. Statist.* **52** (1996), 129–140).

(11) Nam, E. and Rosalsky, A., On the convergence rate of series of independent random variables, In: E. Brunner and M. Denker (eds.), *Research Developments in Probability and Statistics: Festschrift in Honor of Madan L. Puri on the Occasion of his 65th Birthday,* pp. 33–44, VSP International Science Publ., Utrecht, The Netherlands, 1996.

(12) Nam, E., Rosalsky, A. and Volodin, A.I., On convergence of series of independent random elements in Banach spaces, *Stochastic Anal. Appl.,* **17** (1999), 85–97.

(13) Rosalsky, A. and Rosenblatt, J., On the rate of convergence of series of Banach space valued random elements, *Nonlinear Anal.-Theory, Methods Appl.,* **30** (1997), 4237–4248.

(14) Sung, S.H. and Volodin, A.I., On convergence of series of independent random variables, *Bull. Korean Math. Soc.,* **38:4** (2001), 763–772.