Fractal Interest Rate Model

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ABSTRACT. Empirical findings on interet rate dynamics imply that short rates show some long memories and non-Markovin. It is well-known that fractional Brownian motion(fBm) is a proper candidate for modelling this empirical phenomena. fBm, however, is not a semimartingale process. For this reason, it is very hard to apply such processes for asset price modelling. With some modifications, this paper investigate the fBm interest rate theory, and obtain a pure discount bond price and Greeks.

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Key Word : Fractional Brownian motion, Term Structure of Interest Rate, Greeks

1. INTRODUCTION

This paper studies the theoretical aspects of long range dependence in the interest rate model. The short term interest rate is an important factor in many different areas of economics and financial theory. It has strong implications for the pricing of fixed income securities and interest rate derivatives such as bonds, options, and swaps. It is therefore of great interest to obtain a suitable model describing its dynamics.. Diffusion processes are widely used for this purpose. It is, however, still an open debate which of many proposed models is the most adequate when fitting to interest rate data. There is a large number of papers where different models of the term structure are implemented using historical interest rate data.. Here, we focus on the class of single-factor models of the short rate. No economic theory set restrictions on the types of drift and diffusion term , except that the model should be positive. Among these, Ait-Sahalia (1999) fitted some parametric class of models to data. His conclusion was that none of these could be accepted as the true model. Gil-Bazo and Rubio (?) also studied similar models from a time-series nonparametric perspective. They reject a single-factor Markovian model, although conclusions are sensitive to choice of additional conditioning factors. Together with these empirical evidences of non-Markovian short rate, Cajueiro and Tabak present empirical evidence of fractional dynamics in interest rate for different maturities for Brazil bond market. Their results suggest that Brazilian interest rates possess long-range dependence in mean and volatility, and that the degree of long range dependence is time varying.

All these empirical findings on interest rate dynamics imply that short rates show some long memories and non-Markovian.. It is well-known that fractional Brownian motion(fBm) is a proper candidate for modelling this empirical phenomena. fBm, however, is not a semimartingale process. For this reason, it is very hard to apply such processes for asset price modelling. With some modifications, this paper investigate the fBm interest rate theory, and obtain a pure discount bond price and Greeks

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2. Model

We start Vasicek type model. Under P, we assume that short rate follows

$$dr(t) = a(\bar{r} - r(t))dt + \sigma \mathrm{d}B_t^H \tag{1}$$

where we denote by B_t^H the one parameter fractional Brownian motion(fBm) with Hurst parameter $H \in (0,1)$. fBm is the Gaussian process $B_t^H = B_H(t,w), t \in$ R,$w \in \Omega$ satisfying

$$B_0^H = E[B_t^H] = 0$$

for all t and

$$E[B_s^H B_t^H] = \frac{1}{2}\left[|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right]$$

where the expectation is taken under the probability measure P and $(\Omega, \mathcal{F})$ is a measurable space.

In order to apply fBm to interest rate model, we need a stochastic calculus., Note that fBm is not a semimartingale except that H $= \frac{1}{2}$. So we cannot use the theory of stochastic calculus for semimartingale on B_t^H. In this paper, we follow Wick-Ito integral as Okesendal (2003). The integral is denoted by

$$\int_0^T \phi(t,w)\delta B_t^H$$

where the above integral is so called Skorohod or Wick-Ito integral, and we may denote this by

$$\int_0^T \phi(t,w)\delta B_t^H \doteq \int_0^T \phi(t,w)\delta B_t^H = \lim_{\triangle t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \diamond (B_{t_{k+1}} - B_{t_k})$$

where $\diamond$ denotes the Wick product. Note that the difference between this integral and the ordinary integral (pathwise or forward integral) is the use of the Wick product instead of usual product. Then , we have

$$E\left[\int_0^T \phi(t,w)\delta B_t^H\right] = 0$$

if the integral belongs to L^2(P). Following the method of Okesendal (2003), we redefine the no arbitrage :

Definition : A Wick-Skorohod admissibleportfolio $\theta(t)$ is called a strong arbitrage if the corresponding total wealth process $V^\theta(t)$ satisfies

$$V^\theta(0) = 0$$

$$V^\theta(T) \geq 0, \text{a.s. } P$$

$$P[V^\theta(T) > 0] > 0$$

Then we can derive the bond pricing under the no strong arbitrage framework, although fBm is not semimartingale.

The strong solution of (1) is

$$r(t) = (1 - e^{-at})\bar{r} + e^{-at}r(0) + \sigma e^{-at}\int_0^t e^{av}\mathrm{d}B_v^H \tag{2}$$

The price of the pure discount is

$$\begin{aligned} P(t,T) &= \mathsf{E}^{\mathsf{Q}}\left[\exp\left(-\int_t^T r(u)\mathrm{d}u\right)|\mathcal{F}_t\right] \\ &= \mathsf{E}^{\mathsf{P}}\left[\exp\left(-\int_t^T r(u)\mathrm{d}u\right)\frac{\mathsf{dQ}}{\mathsf{dP}}|\mathcal{F}_t\right] \end{aligned}$$

where the Radon-Nikodym derivative should be properly defined. Once the change of measure is determined, then the bond price is as follows:

$$P(t,T) = \mathsf{E}^{\mathsf{P}}\left[\exp\left(-\left(\int_t^T (1-e^{-a(u-t)})\bar{r} + e^{-a(u-t)}r(t) + \sigma e^{-au}\int_t^u e^{av}\mathrm{d}B_v\right)\mathrm{d}u\right)\frac{\mathsf{dQ}}{\mathsf{dP}}|\mathcal{F}_t\right]$$

By little calculation and Fubini, the bond price is given by

$$\begin{aligned} P(t,T) &= \exp\{-\bar{r}(T-t) + \frac{(e^{-aT}-e^{-at})(ar(t)+\bar{r}(2e^{-at}-1) - r(0)(e^{-at}+ae^{-at}))}{ae^{-at}}\} \\ &\quad \times \mathsf{E}^{\mathsf{P}}\left[\exp\left(\int_t^T (\frac{\sigma e^{-a(T-v)}-\sigma}{a})\mathrm{d}B_v^H\right)\frac{\mathsf{dQ}}{\mathsf{dP}}|\mathcal{F}_t\right] \end{aligned}$$

3. Change of Measure

Even though fBm is not a semimartingale, we can associate it with so called fundamental martingale. It is well-known that such a fundamental martingale has the same natural filtration as that of fBm. To define a proper Radon-Nikodym derivative, we introduce some notations. Define, for $0 < s < t$

$$k_H = 2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})$$

$$K_H(t,s) = k_H^{-1}s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}$$

$$\lambda_H = \frac{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$$

$$w_t^H = \lambda_H^{-1}t^{2-2H}$$

and

$$M_t^H = \int_0^t K_H(t,s)dB_s^H,\ t \geq 0$$

where M^H is a Gaussian martingale, called the fundamental martingale, and its quardratic variation

$$\langle M_t^H, M_t^H\rangle = w_t^H$$

The natural filtration of M^H coincides with the natural filtration of the fBm. Note that

$$\mathcal{F} = \mathcal{F}_t^{B^H} = \mathcal{F}_t^{M^H}$$

Furthermore, the stochastic integral with respect to fBm can be represented in terms of the stochastic integral with respect to the fundamental martingale as follows:

$$\int_0^t \phi(s) dB_s^H = \int_0^t K_H^\phi(t,s) dM_s^H$$

where

$$K_H^\phi(t,s) = -2H \frac{d}{ds} \int_s^r \phi(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{1}{2}} dr, \; 0 \leqslant s \leqslant t.$$

Once we obtain the fundamental martingale, then there are several choices of change of measures. Among these, we follow the easiest way as in Norros and Valkeila (?). Denote by

$$\epsilon_t(M^H) = \exp\left(M_t^H - \langle M_t^H, M_t^H \rangle\right) \tag{3}$$

By a standard result of a martingale theory, $\epsilon_t(M)$ is a martingale with expectation 1 under the probability measure P.

4. Bond Pricing

Theorem 1. When the interest rate follows (1) and the Radon-Nikodym derivative has a form as equation (3), then time t price of a discount bond that promises to pay one unit currency at maturity T is

$$P(t,T) = \exp\left(A(t,T) + G(t) + B(t,T) r(t)\right)$$

where

$$A(t,T) = \exp\left(-\bar{r}(T-t) + \frac{\left(e^{-aT} - e^{-at}\right)\left(\bar{r} - r(0)\right) - \left(e^{-a(T-t)} - 1\right)\left((1-e^{-at})\bar{r} + e^{-at} r(0)\right)}{a}\right)$$
$$\times \exp\left(F(t,T) + \lambda_H^{-1}\left(T^{3-2H} - t^{3-2H}\right)\right)$$

,

$$F(t,T) = \frac{1}{2\lambda(H)} \int_t^T \left(K_H^\phi(T,s) s^{1-H}\right)^2 ds$$

,

$$\phi(u) = \frac{\sigma e^{-a(T-t)} - \sigma}{a} + 1$$

$$B(t,T) = \frac{\left(e^{-a(T-t)} - 1\right)}{a}$$

and

$$G(t) = \exp\left(\int_0^t \left(K_H^\phi(T,s) - K_H^\phi(t,s)\right) dM_s^H\right)$$

Proof: see the Appendix 1.

As seen in the bond pricing formula, the function $G(t)$ affect the dynamics of bond price with some memory!

5. Calculating Greeks

It is usually very hard to compute the pricing of interest rate derivatives. For this reason, market practitioners have difficulties to find suitable Greeks of the derivative for hedging. This section investigates the theory, and calculate the Greeks of general bond derivative where the bond pricing formula is known. Recently Malliavin calculus is widely used for calculating Greeks. The basic framework can be explained in a simple way. Let X and Y are two random variables. Then following relationship holds:

$$E\left[f'(x)Y\right] = E\left[[f(X)H\right] \tag{4}$$

where H is some new random variable. The general valuation formula $\Psi(t)$ for bond derivative is expressed as follows

$$\begin{aligned}\Psi(P(t,u)) &= E^{Q}\left[\exp\left(\int_t^T r_s ds\right)\Psi\left(P(T,u)\right)\right],\ t<T<u \\ &= E^{Q_T}\left[\Psi\left(P(T,u)\right)\right]\end{aligned}$$

where $\Psi\left(.\right)$ is the payoff function of the underlying bond price such as bond option, Q_T is the proper forward measure, and .

$$P(T,u) = \exp\left(A(T,u) + G(T) + B(T,u)r(T)\right)$$

$$r(T) = (1-e^{-aT})\overline{r} + e^{-aT}r(0) + \sigma e^{-aT}\int_0^T e^{av}\mathrm{d}B_v^H$$

Now we compute Delta, Delta for bond is defined as:

$$\begin{aligned}Delta &= \frac{\partial}{\partial r_0}E^{Q_T}\left[\Psi\left(P(T,u)\right)\right] \\ &= E^{Q_T}\left[\frac{\partial P(t,u)}{\partial r_0}\Psi'\left(P(T,u)\right)\right] \\ &= E^{Q_T}\left[P(T,u)D^*\left(\frac{P(T,u)}{\int_0^T D_sP(s,u)ds}\right)\right]\end{aligned}$$

Note that H in equation (4) is identified in this case as

$$\begin{aligned}H &= D^*\left(\frac{P(T,u)}{\int_0^T D_sP(s,u)ds}\right) \\ &= D^*\left(P(T,u)M\frac{e^{-aT}-e^{-2aT}}{aP(T,u)}\right)\end{aligned} \tag{5}$$

where D is Malliavin derivatives and D^* is its adjoint operator, called Skorohod integral,

$$D^*(u) = \int_0^T u_t dB_t^H$$

$$D_sP(s,u) = MD_s^HP(s,u)$$

$$\begin{aligned} Mf(x) &= C_H \int_{\mathsf{R}} \frac{f(x-t)-f(x)}{|t|^{3/2-H}} dt, \text{ for } 0 < H < \frac{1}{2} \\ &= C_H \int_{\mathsf{R}} \frac{f(x)}{|t|^{3/2-H}} dt, \text{ for } \frac{1}{2} < H < 1 \\ &= f(x), \text{ for } H = \frac{1}{2} \end{aligned}$$

and note that

$$\begin{aligned} D_s^H P(s,u) &= e^{-aT} P(T,u) D_s^H \left(\sigma e^{-aT} \int_0^T e^{av} \mathrm{d}B_v^H \right) \\ &= \sigma e^{-a(2T-s)} P(T,u) \end{aligned}$$

So one can check the $H = \frac{1}{2}$, where equation (5) is reduced to Brownian case.. Other Greeks can be calculated in a similar way.

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