Slope Rotatability of Icosahedron and Dodecahedron Designs[1)]

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Abstract

Icosahedron and dodecahedron designs are experimental designs which can be used for response surface analysis for the case when three independent variables are involved. When we are interested in estimating the slope of a response surface, slope rotatability is a desirable property. In this paper, we have obtained conditions for icosahedron and dodecahedron designs to have slope rotatability. We also have applied Park and Kim(1992)'s measure of slope rotatability to icosahedron and dodecahedron designs, and observed resultant facts.

Keywords: slope rotatability, icosahedron, dodecahedron

1. Introduction

Icosahedron and dodecahedron designs are response surface designs which can be used for the case when three independent variables are under consideration. One of the advantages of these types of design is that they conveniently afford rotatable designs. They also provide uniform precision designs and orthgonal or near-orthogonal designs if the numbers of center points are suitably determined.

When we are interested in estimating the slope of a response surface, slope rotatability is a desirable property. There are two types of slope rotatability: slope rotatability over axial directions (See Hader and Park (1978).) and slope rotatability over all directions. (See Park (1987).) It was found that every icosahedron design and every dodecahedron design have slope rotatability over all directions. So, from now on, slope rotatability means slope rotatability over axial directions.

In this paper, we will obtain conditions for icosahedron and dodecahedron designs to be slope-rotatable designs. We will also apply the measure of slope rotatability proposed by Park and Kim (1992) to icosahedron and dodecahedron designs, and observe some resultant facts.

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2. A measure of slope rotatability

We consider the second order polynomial regression model

$$y_u = \beta_0 + \sum_{i=1}^{k} \beta_i x_{iu} + \sum_{i=1}^{k} \beta_{ii} x_{iu}^2 + \sum_{i<j}^{k} \beta_{ij} x_{iu} x_{ju} + \varepsilon_u \qquad (u=1,2,\cdots,N),$$

where ε_u's are uncorrelated random errors with mean zero and variance σ^2. The fitted equation by the least squares method can be written as

$$\hat{y} = b_0 + \sum_{i=1}^{k} b_i x_i + \sum_{i=1}^{k} b_{ii} x_i^2 + \sum_{i<j}^{k} b_{ij} x_i x_j.$$

Let us consider a particular class of response surface designs that satisfy the following conditions:

$$\begin{aligned} &Cov(b_i, b_{ii}) = Cov(b_i, b_{ij}) = Cov(b_{ii}, b_{ij}) = Cov(b_{ij}, b_{il}) = 0 \quad (i \neq j \neq l \neq i). \\ &Var(b_i) \text{ are equal for all } i. \\ &Var(b_{ii}) \text{ are equal for all } i. \qquad (2.1) \\ &Var(b_{ij}) \text{ are equal for all } (i,j) \text{ where } i \neq j. \end{aligned}$$

Icosahedron and dodecahedron designs belong to this class.

Park and Kim (1992) proposed a measure of slope rotatability for second order response surface designs. The measure generally has a very complicated form, but for the class of designs satisfying (2.1) it has a much simpler form as follows:

$$\begin{aligned} Q_k(D) &= \frac{1}{\sigma^4} \{4 Var^{(a)}(b_{11}) - Var^{(a)}(b_{12})\}^2 \\ &= \frac{[ii]^4}{\sigma^4} \{4 Var^{(b)}(b_{11}) - Var^{(b)}(b_{12})\}^2, \end{aligned}$$

where $^{(a)}$ and $^{(b)}$ represent 'after scaling' and 'before scaling', respectively, and $[ii] = \sum_{u=1}^{N} x_{iu}^2 / N$ (before scaling). Here scaling means letting the designs have the following first and pure second design moments so that fair comparisons can be made (See Myers (1976, p.135).):

$$[i] = \frac{1}{N} \sum_{u=1}^{N} x_{iu} = 0,$$

$$[ii] = \frac{1}{N} \sum_{i=1}^{N} x_{iu}^2 = 1.$$

The design D is slope-rotatable if and only if the value of $Q_k(D)$ is zero, and D becomes further from a slope-rotatable design as $Q_k(D)$ becomes larger.

3. Icosahedron design

This type of design is for the case of three independent variables. It consists of twelve vertices of the icosahedron $(0, \pm a_1, \pm a_2)$, $(\pm a_2, 0, \pm a_1)$, $(\pm a_1, \pm a_2, 0)$ plus $n_0 \geq 1$ center points. The moments of this configuration are given by

$$[ii] = 4(a_1^2 + a_2^2)/(12 + n_0) \text{ for all } i,$$

$$[iiii] = \sum_{u=1}^{N} x_{iu}^4/N = 4(a_1^4 + a_2^4)/(12 + n_0) \text{ for all } i,$$

$$[iijj] = \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2/N = 4a_1^2 a_2^2/(12 + n_0) \text{ for any } (i, j) \text{ where } i \neq j,$$

and all odd moments $= 0$. $a_1/a_2 = (\sqrt{5}+1)/2 \fallingdotseq 1.6180$ gives an icosahedron design which is rotatable in the Box-Hunter (1957) sense.

The variances of the quadratic coefficients are found to be

$$Var^{(b)}(b_{12}) = \sigma^2/4a_1^2 a_2^2,$$

$$Var^{(b)}(b_{11}) = \frac{(4+n_0)(a_1^2+a_2^2)^2 - (12+n_0)a_1^2 a_2^2}{4n_0(a_1^4 - a_1^2 a_2^2 + a_2^4)(a_1^2+a_2^2)^2} \cdot \sigma^2.$$

Therefore, we have by Eq. (2.4)

$$Q_3(D) = \left\{\frac{4(t^2+1)}{12+n_0}\right\}^4 \left\{\frac{(4+n_0)(t^2+1)^2 - (12+n_0)t^2}{n_0(t^4 - t^2+1)(t^2+1)^2} - \frac{1}{4t^2}\right\}^2,$$

where $t = a_1/a_2$. $Q_3(D)$ depends on a_1 and a_2 only through a_1/a_2. Setting $Q_3(D) = 0$ leads to a fourth degree polynomial equation in t^2:

$$n_0 t^8 - (16+3n_0)t^6 + (16-4n_0)t^4 - (16+3n_0)t^2 + n_0 = 0.$$

The solution of this equation gives the value of t which makes the icosahedron design slope-rotatable. These values of t for various n_0 are given in Table 3.1. From Table 3.1, we note that the value of t that makes the icosahedron design slope-rotatable decreases as n_0 increases. On the other hand, Table 3.2 gives the values of $Q_3(D)$ for the icosahedron designs for various values of t and n_0. It is observed from Table 3.2 that for a given n_0, as t increases ($t \geq 1.0$), the value of $Q_3(D)$ decreases to zero and increases thereafter. The results in Table 3.2 are displayed in Figure 3.1.

Table 3.1. Values of $t = a_1/a_2$ for slope-rotatable icosahedron designs

n_0	1	2	3	4	5	6	7	8	9	10
t	4.2900	3.2744	2.8796	2.6711	2.5433	2.4573	2.3957	2.3496	2.3137	2.2850

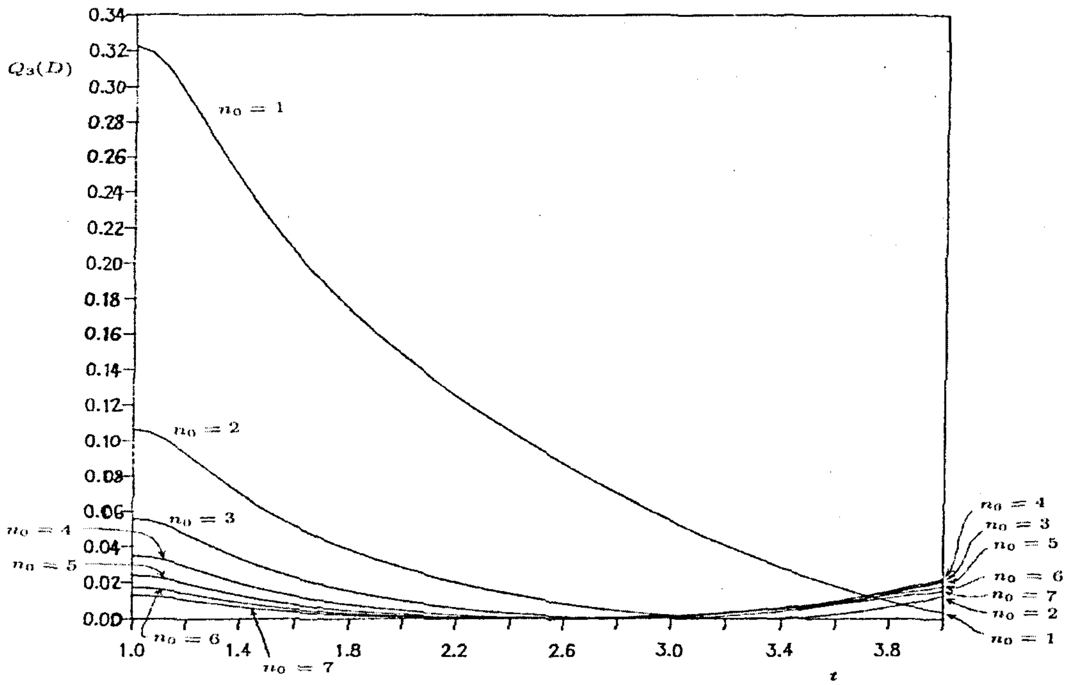

Figure 3.1. Plot of $Q_3(D)$ against $t = a_1/a_2$ for icosahedron designs

Table 3.2. Values of $Q_3(D)$ for icosahedron designs

t n_0	1	2	3	4	5	6	7
1.0	0.3227	0.1066	0.0562	0.0352	0.0240	0.0173	0.0130
1.1	0.3142	0.1024	0.0535	0.0333	0.0227	0.0163	0.0122
1.2	0.2942	0.0927	0.0474	0.0291	0.0196	0.0140	0.0104
1.3	0.2702	0.0812	0.0403	0.0242	0.0161	0.0113	0.0083
1.4	0.2464	0.0701	0.0336	0.0197	0.0128	0.0089	0.0065
1.5	0.2247	0.0602	0.0277	0.0157	0.0100	0.0068	0.0049
1.6	0.2055	0.0518	0.0228	0.0125	0.0077	0.0052	0.0036
1.7	0.1885	0.0445	0.0186	0.0098	0.0059	0.0038	0.0026
1.8	0.1733	0.0383	0.0152	0.0076	0.0044	0.0028	0.0018
1.9	0.1597	0.0329	0.0122	0.0058	0.0032	0.0019	0.0012
2.0	0.1472	0.0281	0.0097	0.0043	0.0022	0.0013	0.0008
2.1	0.1357	0.0238	0.0076	0.0031	0.0015	0.0008	0.0004
2.2	0.1249	0.0200	0.0057	0.0021	0.0009	0.0004	0.0002

2.3	0.1147	0.0166	0.0042	0.0013	0.0004	0.0001	0.0000
2.4	0.1050	0.0135	0.0029	0.0007	0.0002	0.0000	0.0000
2.5	0.0957	0.0107	0.0018	0.0003	0.0000	0.0000	0.0001
2.6	0.0868	0.0082	0.0010	0.0000	0.0000	0.0001	0.0002
2.7	0.0782	0.0061	0.0004	0.0000	0.0002	0.0004	0.0004
2.8	0.0700	0.0042	0.0001	0.0002	0.0005	0.0007	0.0008
2.9	0.0622	0.0027	0.0000	0.0005	0.0010	0.0012	0.0012
3.0	0.0546	0.0015	0.0002	0.0011	0.0016	0.0018	0.0018
3.1	0.0475	0.0006	0.0007	0.0019	0.0025	0.0026	0.0025
3.2	0.0407	0.0001	0.0015	0.0030	0.0035	0.0035	0.0033
3.3	0.0343	0.0000	0.0026	0.0043	0.0048	0.0046	0.0043
3.4	0.0283	0.0003	0.0040	0.0059	0.0062	0.0059	0.0054
3.5	0.0228	0.0011	0.0058	0.0077	0.0079	0.0074	0.0066
3.6	0.0177	0.0023	0.0080	0.0099	0.0099	0.0091	0.0080
3.7	0.0133	0.0041	0.0106	0.0124	0.0121	0.0109	0.0096
3.8	0.0093	0.0064	0.0137	0.0153	0.0146	0.0130	0.0114
3.9	0.0060	0.0093	0.0172	0.0185	0.0174	0.0154	0.0134
4.0	0.0034	0.0127	0.0212	0.0222	0.0205	0.0180	0.0155

4. Dodecahedron design

This type of design is also for the case of three independent variables. It consists of twenty vertices of the dodecahedron $(0, \pm c^{-1}, \pm c)$, $(\pm c, 0, \pm c^{-1})$, $(\pm c^{-1}, \pm c, 0)$, $(\pm 1, \pm 1, \pm 1)$, plus $n_0 \geq 1$ center points. The moments of this configuration are

$$[ii] = 4(c + c^{-1})^2/(20 + n_0) \text{ for all } i,$$

$$[iiii] = 4(c^2 + c^{-2})^2/(20 + n_0) \text{ for all } i,$$

$$[iijj] = 12/(20 + n_0) \text{ for any } (i, j) \text{ where } i \neq j,$$

and all odd moments $= 0$. $c = (\sqrt{5} + 1)/2 \fallingdotseq 1.6180$ gives a dodecahedron design which is rotatable in the Box-Hunter (1957) sense.

The variances of the quadratic coefficients are found to be

$$Var^{(b)}(b_{12}) = \sigma^2/12,$$

$$Var^{(b)}(b_{11})=\frac{(20+n_0)(c^4+c^{-4}+5)-8(c+c^{-1})^4}{4(c^4+c^{-4}-1)\{(20+n_0)(c^4+c^{-4}+8)-12(c+c^{-1})^4\}}\cdot\sigma^2.$$

Hence by Eq. (2.4) it is obtained that

$$Q_3(D)=\left\{\frac{4(t+2)}{20+n_0}\right\}^4\left[\frac{(12+n_0)t^2-32t+(28+3n_0)}{(t^2-3)\{(8+n_0)t^2-48t+6(12+n_0)\}}-\frac{1}{12}\right]^2,$$

where $t=c^2+c^{-2}$.

Setting $Q_3(D)=0$ leads to a fourth degree polynomial equation in t :

$$(8+n_0)t^4-48t^3-(96+9n_0)t^2+528t-(552+54n_0)=0.$$

The solution of this equation gives the value of c which makes the dodecahedron design slope-rotatable. Table 4.1 shows these values of c for various n_0. It is noted from Table 4.1 that the value of c that makes the dodecahedron design slope-rotatable decreases as n_0 increases. The values of $Q_3(D)$ for the dodecahedron designs are given in Table 4.2 for various values of c and n_0. From Table 4.2, we observe that when $n_0\geq 2$, as c increases ($c\geq 1.0$), the value of $Q_3(D)$ decreases to zero and increases thereafter, but when $n_0=1$, $Q_3(D)$ has a local minimum and a local maximum before it becomes zero ($Q_3(D)=0.2017$ and 0.2098 when $c=1.4$ and 1.5, respectively), and decreases to zero and increases thereafter. The results in Table 4.2 are displayed in Figure 4.1.

Table 4.1. Values of c for slope-rotatable dodecahedron designs

n_0	1	2	3	4	5	6	7	8	9	10
c	2.4050	2.3103	2.2362	2.1779	2.1317	2.0948	2.0648	2.0403	2.0199	2.0028

Table 4.2. Values of $Q_3(D)$ for dodecahedron designs

c n_0	1	2	3	4	5	6	7
1.0	0.3185	0.1999	0.1443	0.1111	0.0887	0.0725	0.0602
1.1	0.2806	0.1686	0.1191	0.0904	0.0715	0.0580	0.0480
1.2	0.2269	0.1211	0.0805	0.0589	0.0454	0.0362	0.0295
1.3	0.2018	0.0901	0.0547	0.0379	0.0281	0.0218	0.0174

1.4	0.2017	0.0729	0.0399	0.0258	0.0183	0.0137	0.0106
1.5	0.2098	0.0621	0.0306	0.0185	0.0124	0.0089	0.0067
1.6	0.2047	0.0525	0.0236	0.0133	0.0085	0.0058	0.0042
1.7	0.1742	0.0418	0.0175	0.0093	0.0056	0.0037	0.0025
1.8	0.1274	0.0304	0.0121	0.0060	0.0034	0.0021	0.0013
1.9	0.0816	0.0198	0.0074	0.0033	0.0017	0.0009	0.0005
2.0	0.0466	0.0112	0.0037	0.0014	0.0006	0.0002	0.0001
2.1	0.0234	0.0050	0.0012	0.0003	0.0000	0.0000	0.0000
2.2	0.0095	0.0013	0.0001	0.0000	0.0002	0.0003	0.0004
2.3	0.0023	0.0000	0.0003	0.0007	0.0010	0.0012	0.0012
2.4	0.0000	0.0009	0.0019	0.0025	0.0027	0.0028	0.0027
2.5	0.0017	0.0039	0.0050	0.0054	0.0054	0.0051	0.0048
2.6	0.0068	0.0092	0.0099	0.0097	0.0092	0.0084	0.0077
2.7	0.0154	0.0169	0.0167	0.0156	0.0143	0.0129	0.0116
2.8	0.0276	0.0275	0.0258	0.0234	0.0210	0.0187	0.0166
2.9	0.0440	0.0414	0.0376	0.0335	0.0297	0.0262	0.0231
3.0	0.0651	0.0592	0.0527	0.0464	0.0407	0.0356	0.0313
3.1	0.0918	0.0816	0.0716	0.0624	0.0544	0.0474	0.0414
3.2	0.1251	0.1094	0.0950	0.0823	0.0713	0.0620	0.0540
3.3	0.1661	0.1437	0.1239	0.1067	0.0922	0.0798	0.0694
3.4	0.2164	0.1856	0.1591	0.1366	0.1176	0.1016	0.0882
3.5	0.2774	0.2365	0.2018	0.1727	0.1484	0.1280	0.1109
3.6	0.3509	0.2978	0.2533	0.2163	0.1855	0.1598	0.1383
3.7	0.4393	0.3714	0.3151	0.2685	0.2299	0.1978	0.1710
3.8	0.5447	0.4592	0.3888	0.3308	0.2829	0.2432	0.2101
3.9	0.6700	0.5635	0.4763	0.4048	0.3458	0.2971	0.2565
4.0	0.8183	0.6869	0.5798	0.4922	0.4202	0.3607	0.3113

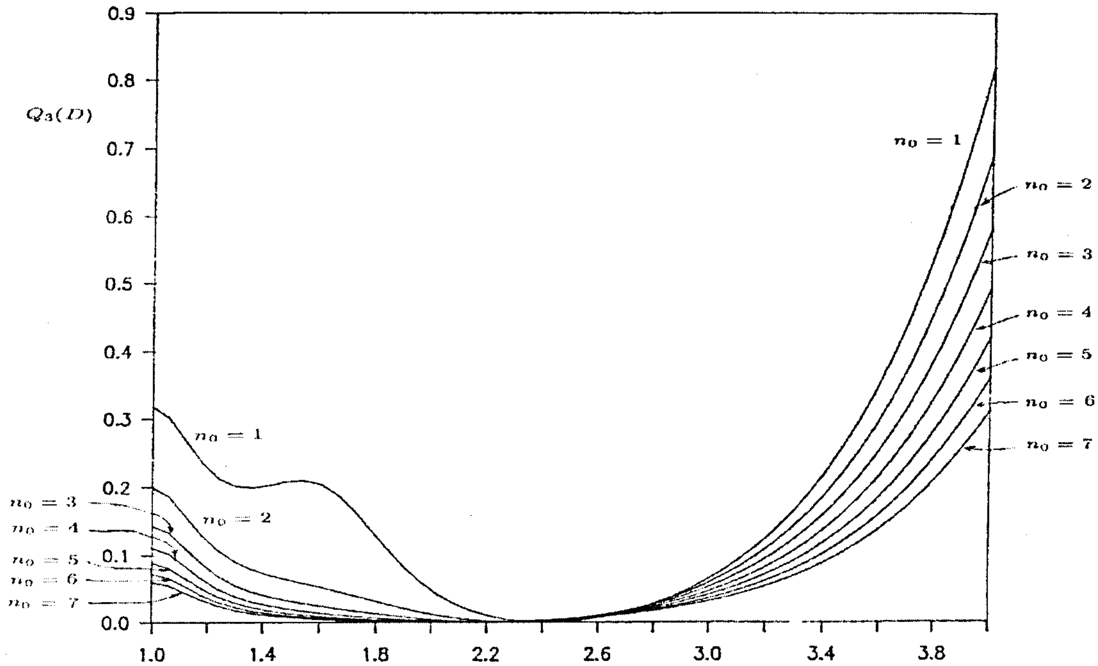

Figure 4.1. Plot of $Q_3(D)$ against c for dodecahedron designs

References

[1] Box, G. E. P. and Hunter, J. S. (1957), "Multifactor experimental designs for exploring response surfaces", *Annals of Mathematical Statistics*, **28**, 195-241.

[2] Hader, R. J. and Park, S. H. (1978), "Slope-rotatable central composite designs", *Technometrics*, **20**, 413-417.

[3] Myers, R. H. (1976), *Response Surface Methodology,* Blacksburg, VA:Author (distributed by Edwards Brothers, Ann Arbor, MI).

[4] Park, S. H. (1987), "A class of multifactor designs for estimating the slope of response surfaces", *Technometrics*, **29**, 449-453.

[5] Park, S. H. and Kim, H. J. (1992), "A measure of slope-rotatability for second order response surface experimental designs", *Journal of Applied Statistics*, **19**, 391-404.