Group Ordering Reference Priors for the Difference of Intraclass Correlation Coefficients in Familial Data

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ABSTRACT

In this paper, we develp the group ordering referecne priors for the differecne of the intraclass correlation coefficient in familial data. Using marginal posterior distributions under those priors, we compare frequentist coverage probabilities.

1. Introduction

The intraclass correlation coefficient ρ is frequently used to measure the degree of intrafamily resemblance with respect to characteristics such as blood pressure, cholesterol, weight, height, stature, lung capacity, and so forth. Statistical inference concerning ρ for a single sample problem based on a normal distribution has been studied by several authors(Rao, 1973 ; Rosner and Donner, 1977, 1979 ; Donner and Bull, 1983 ; Srivastava, 1984 ; Gokhale and SenGupta, 1986 ; Velu and Rao, 1990). Surprisingly, its extension to multisample problems based on several multivariate normal distribution has recevied very little attention.

There is a considerable study of a statistical inference for intraclass correlation coefficient from familial data by several authors. However nothing is known about approach of Bayesian inference except for Kim, Kang, and Lee (2001). But none of the above authors considered any Bayesian inference for unequal family sizes. In practice, we come across families with unequal numbers of children and hence, this is a very important practical problem to consider as a estimation for intraclass correlation coefficient under unequal family sizes. In this thesis, we consider estimation problem for two intraclass correlation coefficients based on two independent multinormal samples under unequal family sizes.

The most frequently used noninformative prior is Jeffreys's prior. But, in spite of its success in one parameter, Jeffreys's prior frequently runs into serous difficulties in the presence of nuisance parameters. To overcome these deficiencies of Jeffreys's priors, Berger and Bernardo (1989, 1992) expounded the reference prior approach of Bernardo (1979) for deriving noninformative priors in multiparameter situations by dividing the parameters into parameters of interest and nuisance parameters. This approach is very successful in various practical problems. As an alternative, we use the method of Peers (1965) to find priors which require the frequentist coverage probability of the posterior

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region of a real-valued parametric function to match the nominal level with a remainder of $o(n^{-\frac{1}{2}})$. These priors, as usually referred to as the first order matching priors.

In the paper, we consider the problem of inferencing ρ_1, ρ_2 using noninformative priors in the following situation :
Suppose we have a sample of measurements from k_1, k_2 families, and let $\boldsymbol{X}_i^*$, $i = 1, 2, \cdots , k_1$ and $\boldsymbol{Y}_j^*$, $j = 1, 2, \cdots , k_2$ represent measurements from the i^{th} family and j^{th} family, respectively , where

$$\boldsymbol{X}_i^* = \left(X_{i1}^*, \cdots, X_{ip_1}^*\right)', \quad \boldsymbol{Y}_j^* = \left(Y_{j1}^*, \cdots, Y_{jp_2}^*\right)' \text{ where } p_1, p_2, k_1, k_2 \geq 2 \ .$$

The structure of the mean vector and the covariance matrix for the familial data is given by Rao(1973) as

$$\boldsymbol{\mu_1^*} = \mu_1 \mathbf{1}_{p_1},\ \Sigma_1^* = \sigma^2(1-\rho_1)I_{p_1} + \rho_1 J_{p_1},\ \ \boldsymbol{\mu_2^*} = \mu_2 \mathbf{1}_{p_2},\ \Sigma_2^* = \sigma^2(1-\rho_2)I_{p_2} + \rho_2 J_{p_2}$$

where $\mathbf{1}_{p_1}$ is a $p_1 \times 1$ vector of 1's and $\mathbf{1}_{p_2}$ is a $p_2 \times 1$ vector of 1's, $\mu_1(-\infty < \mu_1 < \infty)$ and $\mu_2(-\infty < \mu_2 < \infty)$ are the common mean of X_i^*, Y_j^*, respectively, $\sigma^2(\sigma^2 > 0)$ is the common variance of members of the family, and ρ_1, ρ_2 called the intraclass correlation coefficient, which are the coefficient of correlations among the members of the family and $-\frac{1}{p_1-1} < \rho_1 < 1, -\frac{1}{p_2-1} < \rho_2 < 1$, respectively. It is assumed that $\boldsymbol{X}_i^* \sim N_{p_1}(\boldsymbol{\mu_1^*}, \Sigma_1^*), i = 1, 2, \cdots, k_1$, $\boldsymbol{Y}_j^* \sim N_{p_2}(\boldsymbol{\mu_2^*}, \Sigma_2^*), j = 1, 2, \cdots, k_2$ where N_p represents a p-variate normal distribution.

In Section 2, we treat the reparametrization $(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2)$ to $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. In Section 3, we derive, using this transformation, group ordering reference priors when $\eta_1 = \theta_1 - \theta_2$ is the parameter of interest. The sufficient condition for propriety of posterior distributions and marginal posterior densities for η_1 under these priors are given in Section 4.

2. Fisher Information Matrices

Let $\boldsymbol{X}_i = \left(X_{i1}, \cdots, X_{ip_1}\right)' = Q_{p_1 \times p_1}\boldsymbol{X}_i^*$, $\boldsymbol{Y}_j = \left(Y_{j1}, \cdots, Y_{jp_2}\right)' = Q_{p_2 \times p_2}\boldsymbol{Y}_j^*$ where Q is a Helmert orthogonal matrix. Under the orthogonal transformation (1.2), it is obvious that $\boldsymbol{X}_i \sim N_{p_1}(\boldsymbol{\mu_1}, \Sigma_1), i = 1, 2, \cdots, k_1, \boldsymbol{Y}_j \sim N_{p_2}(\boldsymbol{\mu_2}, \Sigma_2), j = 1, 2, \cdots, k_2$, where

$$\boldsymbol{\mu_1} = \left(\sqrt{p_1}\mu_1, 0, \cdots, 0\right)',\ \Sigma_1 = \sigma^2 Diag\left[1 + (p_1 - 1)\rho_1, 1 - \rho_1, \cdots, 1 - \rho_1\right]$$

$$\boldsymbol{\mu_2} = \left(\sqrt{p_2}\mu_2, 0, \cdots, 0\right)',\ \Sigma_2 = \sigma^2 Diag\left[1 + (p_2 - 1)\rho_2, 1 - \rho_2, \cdots, 1 - \rho_2\right]$$

Then the likelihood function of $(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2)$ is $l(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2 \mid \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{(2\pi)^{\frac{p_1k_1+p_2k_2}{2}}}\sigma^{-p_1k_1}(1+(p_1-1)\rho_1)^{-\frac{k_1}{2}}(1-\rho_1)^{-\frac{(p_1-1)k_1}{2}}\sigma^{-p_2k_2}(1+(p_2-1)\rho_2)^{-\frac{k_2}{2}}(1-\rho_2)^{-\frac{(p_2-1)k_2}{2}}$

$$\times e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\mu_1)^2}{1+(p_1-1)\rho_1} + \frac{1}{1-\rho_1}\sum_{m=2}^{p_1}x_{im}^2\right)\right]} e^{-\frac{1}{2\sigma^2}\left[\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\mu_2)^2}{1+(p_2-1)\rho_2} + \frac{1}{1-\rho_2}\sum_{n=2}^{p_2}y_{jn}^2\right)\right]}.$$

Lemma 2.1 (Original Parametrization) The Fisher information matrix of $(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2)$ is given by

$$I_1(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2) = \begin{pmatrix} A & 0 & F & 0 & 0 \\ 0 & B & G & 0 & 0 \\ F & G & E & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & D \end{pmatrix} \tag{2.1}$$

where
$A = \frac{k_1p_1(p_1-1)(1+(p_1-1)\rho_1^2)}{2(1-\rho_1)^2(1+(p_1-1)\rho_1)^2}, B = \frac{k_2p_2(p_2-1)(1+(p_2-1)\rho_2^2)}{2(1-\rho_2)^2(1+(p_2-1)\rho_2)^2}, C = \frac{p_1k_1}{\sigma^2(1+(p_1-1)\rho_1)}, D = \frac{p_2k_2}{\sigma^2(1+(p_2-1)\rho_2)}$,
$E = \frac{k_1p_1+k_2p_2}{2\sigma^4}, F = -\frac{p_1k_1(p_1-1)\rho_1}{2\sigma^2(1-\rho_1)(1+(p_1-1)\rho_1)}, G = -\frac{p_2k_2(p_2-1)\rho_2}{2\sigma^2(1-\rho_2)(1+(p_2-1)\rho_2)}$.

Consider the following transform from $(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2)$ to $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ where
$\rho_1 = \theta_1$, $\rho_2 = \theta_2$, , $\sigma^2 = (1-\theta_1)^{-\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{-\frac{k_1}{k}}(1-\theta_2)^{-\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}}\theta_3, \mu_1 = \theta_4$,and $\mu_2 = \theta_5$ where $k = k_1p_1 + k_2p_2$.

Lemma 2.2 (Reparametrization) The Fisher information matrix for $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ is

$$I_2(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \begin{pmatrix} I_{11} & I_{12} & 0 & 0 & 0 \\ I_{21} & I_{22} & 0 & 0 & 0 \\ 0 & 0 & I_{33} & 0 & 0 \\ 0 & 0 & 0 & I_{44} & 0 \\ 0 & 0 & 0 & 0 & I_{55} \end{pmatrix} \tag{2.2}$$

where
$I_{11} = \frac{k_1p_1(p_1-1)[k+k_1p_1(p_1-1)\theta_1^2]}{2k(1-\theta_1)^2[1+(p_1-1)\theta_1]^2}, I_{12} = I_{21} = -\frac{p_1k_1p_2k_2(p_1-1)(p_2-1)\theta_1\theta_2}{2k(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]}$,
$I_{22} = \frac{k_2p_2(p_2-1)[k+k_2p_2(p_2-1)\theta_2^2]}{2k(1-\theta_2)^2[1+(p_2-1)\theta_2]^2}, I_{33} = \frac{k}{2\theta_3^2}$,
$I_{44} = \frac{p_1k_1}{\theta_3}(1-\theta_1)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{\frac{k_1}{k}-1}(1-\theta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}}$,
$I_{55} = \frac{p_2k_2}{\theta_3}(1-\theta_1)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{\frac{k_1}{k}}(1-\theta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}-1}$.

3. Group Ordering Priors

In this section, we provide, using (2.2), group ordering reference priors for $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ with $\eta_1 = \theta_1 - \theta_2$, parameter of interest, and let $\eta_2 = \theta_2, \eta_3 = \theta_3, \eta_4 = \theta_4, \eta_5 = \theta_5$.

From Berger and Bernardo (1989), we have group ordering reference priors for $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$ when (θ_1, θ_2) is the parameter of interest.

Let $P_1(\boldsymbol{\eta}) = (1-\eta_1-\eta_2)^{\frac{b_1}{2}}(1-\eta_2)^{\frac{b_2}{2}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{b_3}{2}}[1+(p_2-1)\eta_2]^{\frac{b_4}{2}}\eta_3^{\frac{b_5}{2}} \times[Z(\eta_1, \eta_2)]^{\frac{b_6}{2}}B(\eta_1, \eta_2)$.

Theorem 3.1 The group ordering reference priors for $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ are given as follows :
1) With $b_1 = b_2 = b_3 = b_4 = b_6 = 0$, and $b_5 = -4$

$$\pi_R^{(1)}(\boldsymbol{\eta}) \propto P_1(\boldsymbol{\eta}) when(\{\eta_1, \eta_2\}, \{\eta_3, \eta_4, \eta_5\})$$

2) With $b_1 = b_2 = b_3 = b_4 = b_6 = 0$, and $b_5 = -2$

$$\pi_R^{(2)}(\boldsymbol{\eta}) \propto P_1(\boldsymbol{\eta}) when(\{\eta_1, \eta_2, \eta_3\}, \{\eta_4, \eta_5\})$$

3) With $b_1 = b_2 = 0, b_3 = b_4 = -1, b_5 = -2$, and $b_6 = 2$

$$\pi_R^{(3)}(\boldsymbol{\eta}) \propto P_1(\boldsymbol{\eta}) when(\{\eta_1, \eta_2 \eta_4, \eta_5\}, \{\eta_3\})$$

4) With $b_1 = b_2 = b_4 = 0, b_3 = -1, b_5 = -3$, and $b_6 = 1$

$$\pi_R^{(4)}(\boldsymbol{\eta}) \propto P_1(\boldsymbol{\eta}) when(\{\eta_1, \eta_2, \eta_3, \eta_4\}, \{\eta_5\})$$

5) With $b_1 = b_2 = b_3 = 0, b_4 = -1, b_5 = -3$, and $b_6 = 1$

$$\pi_R^{(5)}(\boldsymbol{\eta}) \propto P_1(\boldsymbol{\eta}) when(\{\eta_1, \eta_2, \eta_5\}, \{\theta_3, \theta_4\})$$

6) With $b_1 = b_2 = b_4 = 0, b_3 = -1, b_5 = -3$, and $b_6 = 1$

$$\pi_R^{(4)}(\boldsymbol{\eta}) = \pi_R^{(6)}(\boldsymbol{\eta}) when(\{\eta_1, \eta_2, \eta_4\}, \{\eta_3, \eta_5\})$$

7) With $b_1 = b_2 = b_3 = 0, b_4 = -1, b_5 = -3$, and $b_6 = 1$

$$\pi_R^{(5)}(\boldsymbol{\eta}) = \pi_R^{(7)}(\boldsymbol{\eta}) when(\{\eta_1, \eta_2, \eta_3, \eta_5\}, \{\eta_4\}).$$

4. Marginal Posterior Disributions

Let $P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y}) = \int_{\frac{1}{p_2-1}}^{1} [1 + (p_1 - 1)(\eta_1 + \eta_2)]^{\frac{c_1}{2}} [1 + (p_2 - 1)\eta_2]^{\frac{c_2}{2}} [Z(\eta_1, \eta_2)]^{\frac{c_3}{2}}$ $\times [Z_1(\eta_1, \eta_2)]^{\frac{c_4}{2}} [Z_1(\eta_1, \eta_2) C_1(\eta_1, \eta_2)]^{\frac{c_5 - k}{2}} B(\eta_1, \eta_2) d\eta_2$.

According to Bayes theorem and compute, the marginal posterior distributions of η_1 are given following :
1) With $c_1 = c_2 = 1, c_3 = c_5 = 0$, and $c_4 = -2$, $\pi_R^{(1)} \propto P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y})$
2) With $c_1 = c_2 = c_5 = 1, c_3 = 0$, and $c_4 = -2$, $\pi_R^{(2)} \propto P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y})$
3) With $c_1 = c_2 = c_3 = c_4 = 0$, and $c_5 = 2$, $\pi_R^{(3)} \propto P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y})$
4) With $c_1 = c_4 = 0, c_2 = c_5 = 1$, and $c_3 = -1$, $\pi_R^{(4)} \propto P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y})$
5) With $c_1 = c_5 = 1, c_2 = c_4 = 0$, and $c_3 = -1$, $\pi_R^{(5)} \propto P_2(\eta_1|\boldsymbol{x}, \boldsymbol{y})$

6) With $c_1 = c_4 = 0, c_2 = c_5 = 1$, and $c_3 = -1$, $\pi_R^{(4)} = \pi_R^{(6)}$
7) With $c_1 = c_5 = 1, c_2 = c_4 = 0$, and $c_3 = -1$, $\pi_R^{(5)} = \pi_R^{(7)}$,
where
$B(\eta_1, \eta_2) = \left[k_1 p_1 [1 + (p_2 - 1)\eta_2^2] + k_2 p_2 [1 + (p_1 - 1)\eta_1^2\right]^{\frac{1}{2}}$
$\times[1 + (p_1 - 1)(\eta_1 + \eta_2)]^{-1}[1 + (p_2 - 1)\eta_2]^{-1}(1 - \eta_1 - \eta_2)^{-1}(1 - \eta_2)^{-1}$,
$C_1(\eta_1, \eta_2) = \Big[(1 - \eta_2)[1 + (p_1 - 1)(\eta_1 + \eta_2)][1 + (p_2 - 1)\eta_2] \sum_{i=1}^{k_1} \sum_{m=2}^{p_1} x_{im}^2$
$-(1 - \eta_1 - \eta_2)(1 - \eta_2)[1 + (p_2 - 1)\eta_2] {k_1}^{-1} (\sum_{i=1}^{k_1} x_{i1})^2$
$+(1 - \eta_1 - \eta_2)(1 - \eta_2)[1 + (p_2 - 1)\eta_2] \sum_{i=1}^{k_1} x_{i1}^2$
$+(1 - \eta_1 - \eta_2)[1 + (p_1 - 1)(\eta_1 + \eta_2)][1 + (p_2 - 1)\eta_2] \sum_{j=1}^{k_2} \sum_{n=2}^{p_2} y_{jn}^2$
$-(1 - \eta_1 - \eta_2)(1 - \eta_2)[1 + (p_1 - 1)(\eta_1 + \eta_2)] {k_2}^{-1} (\sum_{j=1}^{k_2} y_{j1})^2$
$+(1 - \eta_1 - \eta_2)(1 - \eta_2)[1 + (p_1 - 1)(\eta_1 + \eta_2)] \sum_{j=1}^{k_2} y_{j1}^2\Big]$,
$Z(\eta_1, \eta_2) = (1 - \eta_1 - \eta_2)^{\frac{(p_1-1)k_1}{k}} [1 + (p_1 - 1)(\eta_1 + \eta_2)]^{\frac{k_1}{k}} (1 - \eta_2)^{\frac{(p_2-1)k_2}{k}}$
$\times[1 + (p_2 - 1)\eta_2]^{\frac{k_2}{k}}$,
$Z_1(\eta_1, \eta_2) = Z(\eta_1, \eta_2)\left\{(1-\eta_1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)](1-\eta_2)[1+(p_2-1)\eta_2]\right\}^{-1}$

The following theorem can be proved by some manipulation.

Theorem 4.1 All the posterior distributions under group ordering reference priors are proper.

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