특이공간으로 구성된 섹터를 이용한 2차 비선형 시스템의 스위칭 제어기 설계

Switching Control for 2nd Order Nonlinear Systems Using Sector Consisting of Singular Hyperplanes

염동회, 임기홍, 최진영
D. H. Yeom, K. H. Im, and J. Y. Choi

Abstract—In this paper, we propose a switching control method for 2nd order nonlinear systems. The main idea behind the method is changing the control law before the trajectory of the solution arrives at the singularities imposed on the denominator of the control law. We show that the control system is asymptotically stable from the fact that the sector consisting of the singular hyperplanes is an invariant set. Illustrative examples are given.

Key Words : Switching, Singular hyperplanes, Sector, Invariant set, 2nd order systems

1. Introduction

In this paper, we propose a switching control scheme for 2nd order nonlinear systems. The proposed method is similar to the inversion-based control such as feedback linearization in respect that some nonlinear dynamics are cancelled by control input and also similar to the constructive Lyapunov function method such as backstepping in respect that the control law is directly induced from the derivative of Lyapunov functions. The difference from the existing methods is that the proposed method is not restricted within a specific class of nonlinear systems. That is, the method does not require the rank conditions and the involutiveness unlike feedback linearization and is applicable to the systems that are not triangular forms contrary to backstepping. In addition, the classification of applicable systems is not clear in sliding mode control. The applicable systems of the proposed method can be discriminated through the observation of the property of the given system on the phase plane.

Each switching control law of the proposed method is designed so as to guarantee the negative definiteness of the time derivative of Lyapunov function. The resulting control input is given in the form of a rational function. So, it is not defined where the denominator of the rational function is equal to zero. To avoid these singularities, the control input is changed by switching rules before the trajectory of the solution meets the singularities. The asymptotic stability of the system can be shown by the invariant property of a sector composed by the singularities.

2. Problem Statements

Consider 2nd order affine nonlinear systems with single input and constant input vector . And suppose that one of the dynamics of the system consists of only a drift term i.e. the control input is not involved in this scalar system.

$$\dot{x} = f(x) + gu \quad \text{or} \quad \begin{cases} \dot{x}_1 = f_1(x) + g_1 u \\ \dot{x}_2 = f_2(x) \end{cases}, \qquad (1)$$

where $x = [x_1 \ x_2]'$. $f(x) = [f_1(x) \ f_2(x)]'$ is a smooth vector field with the property $f(0) = 0$ and $g = [g_1 \ 0]'$ is a constant input vector. At least one of the equilibria of the system is the origin because $f(0) = 0$ and $u(0) = 0$. The objective is the regulation of both states, x_1 and x_2.

Construct a quadratic function $V = x'Mx$ for this system, where M is a symmetric positive definite matrix. Derivation of the function with respect to time along the trajectory of the solution of the system yields

$$\begin{aligned} \dot{V} &= \dot{x}'Mx + x'M\dot{x} \\ &= 2x'M f(x) + 2x'M gu \end{aligned} \qquad (2)$$

because M is a symmetric matrix and u is a scalar. Suppose that the control input is defined as follows

$$u = \frac{1}{x'Mg}\left[-x'M f(x) - x'Mx\right]. \qquad (3)$$

Since $\dot{V} = -2x'Mx < 0$ for any x except the origin by substituting (3) for (2), the origin of the system may be regarded as asymptotically stable. But that is not reasonable because the control input (3) is not defined where the denominator of the rational function $x'Mg$ is equal to zero. Here, we define a terminology what is called singular hyperplane.

Definition 2.1 (Singular Hyperplane) *The set of points where the denominator of the control input (3) is equal to zero is named the singular hyperplane generated by control input (simply singular hyperplane),* S . *That is, for given system (1) with control input (3),*

$$S = \{ x \in \Re^2 \mid x'Mg = 0,\ x \neq 0 \} \quad (4)$$

is the singular hyperplane with Mg as its normal vector. ■

To avoid the situation that the trajectory of the solution meets the singular hyperplane, the control input has to be changed by selecting another M before the trajectory arrives at the singular hyperplane. Unfortunately, the asymptotic stability of the origin is not guaranteed only by avoiding the singular hyperplane because of the discontinuity of Lyapunov functions by replacing M and an unexpected behavior caused by switching action. Hence, there is a need to develop a method which makes the trajectory avoid the singular hyperplane, stabilizes the system in spite of the discontinuity of Lyapunov functions, and overcomes an unexpected behavior caused by switching action, simultaneously.

3. Switching Control Law

In this section, we propose a switching control scheme which stabilizes a class of 2nd order nonlinear systems given in (1) by means of the similar control law given in (3). Consider again a system given in (1) with a discontinuous control input as follows

$$\begin{cases} \dot{x}_1 = f_1(x) + u \\ \dot{x}_2 = f_2(x) \end{cases}$$
$$u = \frac{1}{x'M_k g}\left[-x'M_k f(x) - x'M_k x\right], \quad (5)$$

where g is normalized into [1 0]' to simplify the problem. The control input u differs from (3) in respect that it is not continuous anymore because M_k in (5) is changed according to switching signal. Note that u is well defined around the origin because the convergence rate of the numerator of the control input is faster than that of the denominator from the fact that the orders of the numerator and the denominator with respect to 0 are 2 and 1, respectively. As mentioned in Section 2, u should be changed by switching M_k among appropriate candidates of positive definite matrices before the trajectory of the solution arrives at the singular hyperplane defined in Definition 2.1. The problems are when M_k should be changed and how to determine an appropriate M_k.

The first problem, when M_k should be changed, is concerned with the condition that generates switching signal. When the trajectory is sufficiently close to the current singular hyperplane, one must switch the control input to another one with a new singular hyperplane. The simplest way is let switching occur when the distance between the singular hyperplane and the current state on the trajectory is less than a fixed value ε. In this way, however, there is a problem where the trajectory lies in the ball with radius ε and the origin as center. In this ball, the switching repeats infinitely because the distance from the current state to the singular hyperplane is always less than ε. This problem can be solved by using the ratio of the distance form the current state to the origin and the singular hyperplane. That is, the switching condition is set as follows

$$\frac{|x'M_k g|}{\|x\|_2 \|M_k g\|_2} < \varepsilon . \quad (6)$$

The second problem, how to determine an appropriate M_k, is more complicated because the switching action may lead to unexpected behaviors in the system such as chaotic transient response and even instability [8,9,10,12]. For example, it is well known that a switched system consisting of stable subsystems may become unstable under a certain switching rule [9,15].

In this paper, selecting M_k is directly related to the singular hyperplanes because $M_k g$ is the normal vector of the singular hyperplanes. To go ahead the discussion, we need to define some terminologies called attraction region, zero attraction region, and sector.

Definition 3.1 (Attraction Region) *The region where the product of x_2 and the drift scalar system $f_2(x)$ in (5) is less than zero is called the attraction region. That is,*

$$AR = \{ x \in \Re^2 \mid x_2 f_2(x) < 0 \} \quad (7)$$

is the attraction region for x_2 (simply AR). ■

Definition 3.2 (Zero Attraction Region) *The attraction region in touch with the origin is called the zero attraction region abbreviated to ZAR.* ■

Definition 3.3 (Sector) *The unilateral region between below one singular hyperplane and above the other singular hyper -plane is called the sector.* ■

Note that the region between below or above both singular hyperplanes is not a sector. In case of 2nd order system given in (5), the criterion of whether the current state x belongs to the sector consisting of two singular hyperplanes with $M_1 g$, $M_2 g$ as their normal vectors is

$$x'M_1 g \cdot x'M_2 g < 0 . \quad (8)$$

Using the above definitions, we can classify the system to which the proposed switching method can be applicable as follows

Assumption 3.4 *Suppose that 2nd order nonlinear systems given in (5) have ZAR that include both unilateral sectors at least around the origin.* ■

For example, the proposed method can be applicable to a system whose attraction region is formed as follows.

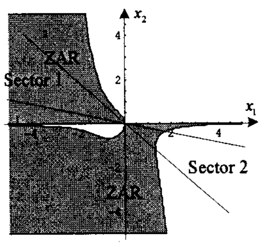

Fig. 3.1. Attraction regions for $\dot{x}_2 = x_1 + x_1^3 x_2 + x_2^2$.

In Fig. 3.1, both attraction regions are the zero attraction region, ZARs include the sector 1 entirely and partially include the sector 2 around the origin. The numerical result for this system is given in Example 1 in Section 5. The other side, the proposed method can not be applicable to a system whose attraction region is formed like as Fig. 3.2.

Observing AR and ZAR of the given system, the sector should be appropriately assigned to satisfy Assumption 3.4. That is, one should select M_k so as to generate suitable singular hyperplanes for applicable systems.

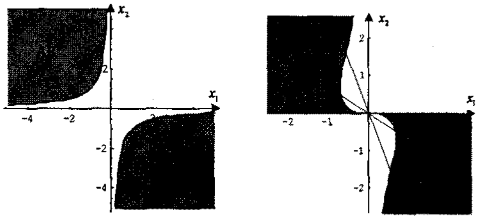

Fig. 3.2. Attraction regions for $\dot{x}_2 = x_2 + x_1x_2^2$ and $\dot{x}_2 = x_2 + x_1^3 + x_1x_2^2$.

4. Stability Analysis

In this section, we discuss the stability issue of 2nd order nonlinear systems with the proposed switching control. If the trajectory stays on a sector on ZAR in the future, we can assert that the origin is asymptotically stable from the invariant property of the sector consisting of the singular hyperplanes. The following theorem shows that a sector consisting of singular hyperplanes is an invariant set when some switching rule is added to (6).

Theorem 4.1 *If the following switching rules are applied to the control system (5)*

$$\frac{|x'M_k g|}{\|x\|_2 \|M_k g\|_2} < \varepsilon \quad \text{or} \quad (x \times \dot{x})'(x \times \vec{S}_k) < 0 \qquad (9)$$

where $k = \{1,2\}$, $M_k = [\alpha_k \ \gamma_k; \ \gamma_k \ \beta_k] > 0$, $\times$ *denote the cross product of vectors, and* $\vec{S}_k$ *denotes the tangent vector of the corresponding singular hyperplane, then the sector consisting of the singular hyperplanes composed by* M_k *is a positively invariant set (Proof is omitted by space limitation).* ■

Remark 4.2 Each singular hyperplane plays the role of a kind of attractor and has a temporary invariant set what is called the Lyapunov level surface. ■

From now, we discuss the stability issues by means of the invariant property of the sector. As mentioned in the introduction of this section, we show that the trajectory of the solution converges to the origin if it stays in the sector on ZAR after a certain time.

Theorem 4.3 (Local Stability) *The origin of the system satisfying Assumption 3.4 is locally asymptotically stabilizable (Proof is omitted by space limitation).* ■

Intuitively speaking, the above theorem states that the terminal point of the trajectory admits of only the origin because the trajectory never escape from the corn shaped sector and x_2 converges to zero on there.

Theorem 4.4 (Globally Asymptotic Stability) *If ZAR include both unilateral sectors entirely, then the origin of the system is globally asymptotically stable (Proof is omitted by space limitation).* ■

5. Conclusions

We proposed a switching control method for 2nd order nonlinear systems. The proposed method can be used more widely in applications comparing with the existing methods such as feedback linearization, back stepping, and sliding mode control. The striking features of the proposed method are intuitive and simple design procedure, and the independency with respect to $f_1(x)$. However, the proposed method can be applied to 2nd order system with single input. The future research will be carried out concerning with this issue to extend the method to nth order systems.

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