Fuzzy quasi extremally disconnected spaces

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Abstract

In this paper, we introduce the concept of fuzzy quasi extremally disconnectedness in fuzzy bitopological space, which is a generalization of fuzzy extremally disconnectedness due to Ghosh [5] in fuzzy topological space and investigate some of its properties using the concepts of quasi-semi-closure, quasi-θ-closure and related notions in a fuzzy bitopological setting.

Key Words: fuzzy bitopological spaces; fuzzy quasi semi-open, fuzzy quasi preopen and fuzzy quasi semi-preopen sets; quasi-semi-closure and quasi-θ-closure; fuzzy quasi extremally disconnected spaces.

1. Introduction and Preliminaries

Motivated by the fact that there are some non-symmetric fuzzy topological structures, Kubiak [8] introduced the notion of fuzzy bitopological space, henceforth fbts for short, (A triple (X,τ_1,τ_2) where X is a non-empty set and τ_1 and τ_2 are fuzzy topology on X is called a fuzzy bitopological space) and initiated the bitopological aspects due to Kelly [7] in the theory of fuzzy topological spaces. Since then several authors [3,4,6,8-10,12] have contributed to the subsequent development of various fuzzy bitoplogical properties. Recently, Park et.al. [12] introduced the concept of fuzzy quasi open (fuzzy quasi closed) sets and studied its basic properties. They also introduced the concepts of fuzzy quasi separation axioms, fuzzy quasi connectedness and fuzzy quasi continuity by using quasi Q-neighborhoods and quasi-neighborhoods.

In this paper, we introduce the concepts of fuzzy quasi semiopen, fuzzy quasi preopen and fuzzy quasi pre-semiopen sets and study their basic properties. We also introduce the concept of fuzzy quasi extremally disconnectedness and characterize its properties in terms of the concepts of quasi-semi-closure, quasi-θ-closure and related notions in fuzzy bitopological setting.

For definitions and results not explained in this paper, we refer to the papers [2,11-13] assuming them to be well known. A fuzzy point in X with support $x \in X$ and value α $(0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A of X, $1-A$ will stand for the complement of A. By 0_X and 1_X, we will mean respectively the constant fuzzy sets taking on the values 0 and 1 on X.

A fuzzy set A of a fbts (X,τ_1,τ_2) is called fuzzy quasi-open (briefly, fqo) if for each fuzzy point $x_\alpha \in A$ there exists either a τ_1-fo set U such that $x_\alpha \in U \leq A$, or a τ_2-fo set V such that $x_\alpha \in V \leq A$. Fuzzy quasi-closed (briefly, fqc) set if the complement is a fqo set. A fuzzy set A is said to be quasi-Q-nbd (resp. quasi-nbd) of a fuzzy point x_α if there exists a fqo set U such that $x_\alpha q U \leq A$ (resp. $x_\alpha \in U \leq A$). A fuzzy point x_α belongs to $\mathrm{qcl}(A)$ (the quasi closure of a fuzzy set A with respect to a fuzzy bitopology) if every fqo quasi-Q-nbd of x_α is q-coincident with A [12].

Definition 1.1. Let A be a fuzzy set of a fbts (X,τ_1,τ_2). Then A is called:

(a) fuzzy quasi semiopen (briefly, fqso) if there exists a fqo set B such that $B \leq A \leq \mathrm{qcl}(B)$,

(b) fuzzy quasi semiclosed(briefly, fqsc) if there exists a fqc set B such that $\mathrm{qint}(B) \leq A \leq B$,

(c) fuzzy quasi preopen (briefly, fqpo) if $A \leq \mathrm{qint}(\mathrm{qcl}(A))$,

(d) fuzzy quasi preclosed (briefly, fqpc) if $A \geq \mathrm{qcl}(\mathrm{qint}(A))$,

(e) fuzzy quasi semi-preopen (briefly, fqspo) if $A \leq \mathrm{qcl}(\mathrm{qint}(\mathrm{qcl}(A)))$,
(f) fuzzy quasi semi-preclosed (briefly, fqspc) if $A \geq \mathrm{qint}(\mathrm{qcl}(\mathrm{qint}(A)))$.

Remark 1.2. Every fqo (resp. fqc) set is both fqso (resp. fqsc) and fqpo (resp. fqpc) but the converses are not true. Every fqso (resp. fqsc) set, or fqpo (resp. fqpc) set is fqspo (resp. fqspc) set but the converses are not true.

Example 1.3. Let $X=\{a,b,c\}$ and A_i (i=1,2,3,4) be a fuzzy set of X defined as follows:

$$A_1(a)=0.5, A_1(b)=0.8, A_1(c)=0.5;$$
$$A_2(a)=0.5, A_2(b)=0.3, A_2(c)=0.5;$$
$$A_3(a)=0.6, A_3(b)=0.2, A_3(c)=0.6;$$
$$A_4(a)=0, \quad A_4(b)=0.3, A_4(c)=0.$$

(a) Let $\tau_1=\{1_X, 0_X, A_1\}$ and $\tau_2=\{1_X, 0_X, A_2\}$ be fuzzy topologies on X. Consider a fuzzy set B of X defined $B(a)=0.5, B(b)=0.6, B(c)=0.5$. Then B is a fqso set but neither a fqo set nor a fqpo set.
(b) Let $\tau_1=\{1_X, 0_X, 1-A_1\}$ and $\tau_2=\{1_X, 0_X, 1-A_2\}$ be fuzzy topologies on X. Consider a fuzzy set C of X defined $C(a)=0.5, C(b)=0.4, C(c)=0.5$. Then C is a fqpo set but neither a fqo set nor a fqso set.
(c) Let $\tau_1=\{1_X, 0_X, A_3\}$ and $\tau_2=\{1_X, 0_X, A_4\}$ be fuzzy topologies on X. Consider a fuzzy set D of X defined $D(a)=0.7, D(b)=0.5, D(c)=0.2$. Then D is a fqspo set but neither a fqso set nor a fqpo set.

Theorem 1.4. For a fuzzy set A in a fbts (X,τ_1,τ_2), the following are equivalent:

(a) A is fqsc (resp. fqso) set,
(b) $\mathrm{qint}(\mathrm{qcl}(A)) \leq A$ (resp. $A \leq \mathrm{qcl}(\mathrm{qint}(A))$),
(c) $\mathrm{qint}(A) = \mathrm{qint}(\mathrm{qcl}(A))$ (or, $\mathrm{qcl}(A) = \mathrm{qcl}(\mathrm{qint}(A))$).

Theorem 1.5. (a) Any union of fqso (resp. fqpo, fqspo) sets is a fqso (resp. fqpo, fqspo) set, and
(b) any intersection of fqsc (resp. fqpc, fqspc) sets is a fqsc (resp. fqpc, fqspc) set.

Definition 1.6. Let A be a fuzzy set of a fbts X. Then

(a) $\mathrm{qscl}(A) = \cup\{B \mid B$ is fqsc set, $A \leq B\}$ is called quasi semi-closure of A,
(b) $\mathrm{qsint}(A) = \cup\{B \mid B$ is fqso set, $A \geq B\}$ is called quasi semi-interior of A.

Remark 1.7. For a fuzzy set of fbts X, $\mathrm{qint}(A) \leq \mathrm{qsint}(A) \leq A \leq \mathrm{qscl}(A) \leq \mathrm{qcl}(A)$, $\mathrm{qsint}(1-A) = 1-\mathrm{qscl}(A)$ and $\mathrm{qscl}(A)$ (resp. qsint(A)) is fqsc (resp. fqso) set.

Theorem 1.8. Let A be a fuzzy set of a fbts X. Then $x_\alpha \in \mathrm{qscl}(A)$ if and only if for each fqso set U with $x_\alpha qU$, UqA

2. Fuzzy quasi extremally disconnected spaces

Definition 2.1. A fbts X is said to be fuzzy quasi extremally disconnected (briefly, FQED) if quasi closure of every fqo set is fqo in X.

Theorem 2.2. A fbts X is FQED if and only if any two non-q-coincident fqo sets of fbts X have non-q-coincident quasi-closures.

Theorem 2.3. The following are equivalent for a fbts X

(a) X is FQED.
(b) For each fqso set A of X, $\mathrm{qcl}(A)$ is a fqo set.
(c) For each fqso set A of X, $\mathrm{qscl}(A)$ is a fqo set.
(d) For each fqso set A and each fqso set B with $A\bar{q}B$, $\mathrm{qcl}(A)\bar{q}\,\mathrm{qcl}(B)$.
(e) For each fqso set A of X, $\mathrm{qcl}(A) = \mathrm{qscl}(A)$.
(f) For each fqso set A of X, $\mathrm{qscl}(A)$ is a fqc set.
(g) For each fqsc set A of X, $\mathrm{qint}(A) = \mathrm{qsint}(A)$.
(h) For each fqsc set A of X, $\mathrm{qsint}(A)$ is a fqo set.

Definition 2.4 [12]. A fuzzy point x_α in a fbts X is said to be a quasi-θ-cluster point of a fuzzy set A if for every fqo quasi-Q-nbd U of x_α, $\mathrm{qcl}(U)qA$. The set of all fuzzy quasi-θ-cluster points of A is called the quasi-θ-closure of A and will be denoted by $\mathrm{qcl}_\theta(A)$. A fuzzy set A is called quasi-θ-closed if $A = \mathrm{qcl}_\theta(A)$.

It is easy to see that $\mathrm{qcl}(A) \leq \mathrm{qcl}_\theta(A)$ for any fuzzy set A of a fbts X.

Lemma 2.5. For any fqpo set A of a fbts X, $\mathrm{qcl}(A) = \mathrm{qcl}_\theta(A)$.

Theorem 2.6. The following are equivalent for a fbts X:

(a) X is FQED.
(b) The quasi-closure of every fqspo set of X is

fqo set.

(c) The quasi-θ-closure of every fqpo set of X is fqo set.

(d) The quasi-closure of every fqpo set of X is fqo set.

Lemma 2.7. For a fuzzy set A of a fbts X,

(a) $\mathrm{qint}(\mathrm{qcl}(A)) \leq \mathrm{qscl}(A)$,

(b) $\mathrm{qint}(\mathrm{qscl}(A)) = \mathrm{qint}(\mathrm{qcl}(A))$.

Definition 2.8. Let A be a fuzzy set of a fbts X. Then A is called:

(a) fuzzy quasi regularly open (briefly, fqro) if $A = \mathrm{qint}(\mathrm{qcl}(A))$,

(b) fuzzy quasi regularly closed (briefly, fqrc) if $A = \mathrm{qcl}(\mathrm{qint}(A))$.

Every fqro (resp. fqrc) set is fqo (resp. fqc) set and fqsc (resp. fqso) set but the converse need not be true as following example shows.

Example 2.9. Let $X = \{a, b, c\}$, $\tau_1 = \{1_X, 0_X, A\}$ and $\tau_2 = \{1_X, 0_X, B\}$, where A and B are fuzzy sets in X defined as follows:

$$A(a) = 0.7,\ A(b) = 0.5,\ A(c) = 0.3;$$
$$B(a) = 0.4,\ B(b) = 0.6,\ B(c) = 0.5.$$

Then A is a fqo set but not a fqro set. Also a fuzzy set C of X, defined by

$$C(a) = 0.4,\ C(b) = 0.5,\ C(c) = 0.3,$$

is a fqsc set but not a fqro set.

Lemma 2.10. Let A be a fuzzy set of a fbts X. Then we have

(a) A is fqpo if and only if $\mathrm{qscl}(A) = \mathrm{qint}(\mathrm{qcl}(A))$.

(b) A is fqpo if and only if $\mathrm{qscl}(A)$ is fqro set.

(c) A is fqro set if and only if A is fqpo and fqsc.

Theorem 2.11. In a fbts X, the following are equivalent:

(a) X is FQED.

(b) $\mathrm{qscl}(A) = \mathrm{qcl}_\theta(A)$ for every fqpo (or fqso) set A of X.

(c) $\mathrm{qscl}(A) = \mathrm{qcl}(A)$ for every fqspo set A of X.

Theorem 2.12. In a fbts X, the following are equivalent:

(a) X is FQED.

(b) For each fqspo set A and each fqso set B such that $A\bar{q}B$, $\mathrm{qcl}(A)\bar{q}\ \mathrm{qcl}(B)$.

(c) For each fqpo set A and each fqso set B such that $A\bar{q}B$, $\mathrm{qcl}(A)\bar{q}\,\mathrm{qcl}(B)$.

Theorem 2.13. A fbts X is FQED if and only if every fqso set is fqpo set.

Definition 2.14. Let $f: X \to Y$ be a mapping from a fbts X to another fbts Y. f is called:

(a) fuzzy quasi semi-continuous (briefly, fqsc) if $f^{-1}(V)$ is a fqso set of X for each fqo set V of Y, equivalently, $f^{-1}(V)$ is a fqsc set of X for each fqc set V of Y.

(b) fuzzy quasi almost-open (briefly, fqao) if $f(U)$ is a fqo set of Y for each fqro set U of X.

Lemma 2.15. Let $f: X \to Y$ be a mapping from a fbts X to another fbts Y. Then

(a) f is fqsc if and only if $f(\mathrm{qscl}(A)) \leq \mathrm{qcl}(f(A))$ for each fuzzy set A of X.

(b) f is fqao if and only if $f(\mathrm{qint}(A)) \leq \mathrm{qint}(f(A))$ for each fqsc set A of X.

Lemma 2.16. If $f: X \to Y$ is fqsc and fqao mapping, then $f(A)$ is a fqpo set of Y for each fqpo set A of X.

Lemma 2.17. If $f: X \to Y$ is fqsc and fqao mapping, then we have

(a) $f^{-1}(B)$ is a fqsc set of X for each fqsc set B of Y.

(b) $f^{-1}(B)$ is a fqso set of X for each fqso set B of Y.

Theorem 2.18. Let $f: X \to Y$ be a fqsc and fqao surjection. If X is FQED, then Y is also FQED.

References

[1]. K. K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981), 14-32.

[2].C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[3].N. R. Das and D. C. Baishya, Fuzzy bitopological space and separation axioms, J. Fuzzy Math. 2 (1994), 389-396.

[4]. N. R. Das and D. C. Baishya, On fuzzy open maps, closed maps and fuzzy continuous maps in a fuzzy bitopological spaces, (Communicated).

[5]. B. Ghosh, Fuzzy extremally disconnected spaces, Fuzzy Sets and Systems 46 (1992) 245-250.

[6].A. Kandil, Biproximaties and fuzzy bitopological spaces, Simon Stevin 63 (1989), 45--66.

[7]. J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963), 71-89.

[8]. T. Kubiak, Fuzzy bitopological spaces and fuzzy quasi proximities, Proc. Polish Sym. Interval and Fuzzy Mathematics, Poznan, August (1983), 26-29.

[9]. S. S. Kumar, On fuzzy pairwise α-continuity and fuzzy pairwise pre-continuity, Fuzzy Sets and Systems 62 (1994), 231-238.

[10]. S. S. Kumar, Semi-open sets, semi-continuity and semi-open mappings in fuzzy bitopological spaces, Fuzzy Sets and Systems 64 (1994), 421-426.

[11]. S. Nanda, On fuzzy topological spaces, Fuzzy Sets and Systems 19 (1986), 193-197.

[12]. J. H. Park, J. K. Park and S. Y. Shin, Fuzzy quasi-continuity and fuzzy quasi-separation axioms, International. J. of Fuzzy Logic and Intelligent Systems, 8(1998), 83-91.

[13]. P. M. Pu and Y. M. Liu, Fuzzy topology I. Neighborhood structure of fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980), 571-599.

[14]. C. K. Wong, Fuzzy topology: Product and quotient theorems, J. Math. Anal. Appl. 45 (1974) 512--521.

[15]. H. T. Yalvac, Fuzzy sets and functions on fuzzy spaces, J. Math. Anal. Appl. 126 (1987), 409-423.