제한된 곡률을 갖는 최단경로에 대한 새로운 기하학적 증명

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A New Geometric Proof on Shortest Paths of Bounded Curvature

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요 약

We consider a point robot in the plane whose turning radius is constrained to be at least 1 and that is not allowed to make reversals. Given a *starting configuration* (a location and an orientation) for the robot, we give a new geometric proof on the combinatorial structure of curvature-constrained shortest paths to a final point with free orientation.

1 Introduction

Imagine that a point robot is moving in the plane whose turning radius is constrained to be at least 1. We assume that the robot is moving only forward, that is, it is not allowed to make reversals.

Dubins [3] was perhaps the first to study optimal paths of such robots and proved that a curvature-constrained shortest path from a starting configuration to a final configuration consists of at most 3 segments of types CCC or CSC, or their substrings, where C is a circular arc of unit circle and S specifies a straight line segment.

An interesting variation is finding a curvature-constrained shortest path from a configuration (a point with a fixed orientation θ) to a point with free orientation. In this paper, we give a characterization of a curvature-constrained shortest path from an initial configuration s $= (s, \theta)$ to a final location t that it consists of at most 2 segments. More precisely, it is of type CC when t lies in the interior of two unit discs tangent to s, or of type CS (or its substrings) otherwise. Note that the shortest path consists of one segment only when t lies on the boundary of two unit discs tangent to s or on the half-line from s in direction θ of s.

Theorem 1 *A curvature-constrained shortest path from* s *to a final location* t *is of type* CC *when* t *lies in the interior of two unit discs tangent to* s*, or of type* CS *or its substrings otherwise.*

Before our result, Boissonnat and Bui [2] have proved Theorem 1 using a tool from Control Theory, known as the minimum principle of Pontyagin and transversality condition. Their results have been used to construct optimal paths for car-like robots [4].

Our results are essentially the same as those of Boissonnat and Bui [2], but the interest of our work lies in the method of proof: we make use of only geometry of the curvature-constrained paths (we don't use(need) any black box that gives an answer). The proofs themselves are quite simple and intuitive. Our characterization indeed gives continuous deformation of a Dubins path to another Dubins path (that has a different final orientation) with shorter length, which may shed some light to other related problems.

2 Shortest path to a point

A Greek mathematician Archimedes gave the following axioms on the length of convex curves connecting two distinct points in the plane.

Lemma 2 *For two distinct points p and q,*

(i) The line segment from p to q is shorter than any other path from p to q.
(ii) For two convex paths from p to q, one inside the other, the inside one is the shorter.

In this section we give a characterization of a path from an initial configuration to a final location such that the path is a shortest among all paths with a prescribed curvature bound. We denote by γ a curvature-constrained path from s to a final location t. A subpath of γ from a point x to another y is denoted by $\gamma(x, y)$.

Lemma 3 *No path of type CCC is shortest.*

Proof. Assume to the contrary that a shortest path γ consists of 3 circular arcs. Considering the forward tangent of the path at t as the orientation of the final configuration, γ must be a Dubins shortest path. Dubins [3] showed that the middle (intermediate) arc C_m of γ has length $> \pi$. This implies that we can always draw a line segment from t to a point p in the interior of C_m such that the segment is tangent to C_m at p. We can replace the subpath $\gamma(p, t)$ with the line segment $\overline{pt}$ and get a new C^1 path with smaller length by Lemma 2 (i). (See Figure 1 (a).) □

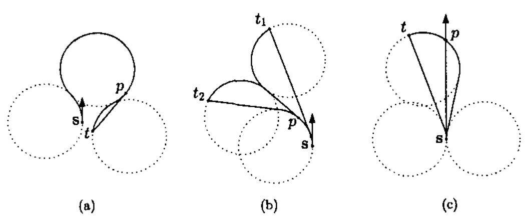

Figure 1: The cases when t lies outside of two discs tangent to s.

From now on, and without loss of generality, we assume that the orientation of s is upward vertical. We denote by D_L (resp. D_R) the left (resp. right) disc tangent to s. We denote by $R(xy)$ a clockwise arc and by $L(xy)$ a counterclockwise arc from x to y of a unit circle through x and y. We also denote by R and L the orientation types of circular arcs, if understood in the context.

We need a technical lemma to show Lemma 5, the first half of Theorem 1.

Lemma 4 *Given $t \notin (D_L \cup D_R)$, every shortest path of type SC, CSC or CCC has the last arc of length at most π.*

Proof. Let γ be a shortest path of such a type whose last arc has length bigger than π, and let D be the unit disc of the last arc. Since $t \notin (D_L \cup D_R)$, we can always slide D along γ backward to s until it stops intersecting t. There exists a Dubins' shortest path from s to t with one of two tangents of D at t such that it has length smaller than γ. □

Lemma 5 *Given $t \notin (D_L \cup D_R)$, a shortest path is of type CS.*

Proof. Assume to the contrary that a shortest path γ is not of type CS. Considering the forward tangent of γ at t as the orientation of the final configuration, γ is a Dubins shortest path. From Lemma 3 γ is not of type CCC.

First consider the case that γ is of type either SL or LSL. Then we can always draw a line through t and tangent to D_L at q such that $L(sq)$ of D_L followed by $\overline{qt}$ form a new C^1 path with smaller length. The case that γ is of type either SR or RSR can be handled symmetrically.

Now consider the case that γ is of type CC, LSR or RSL. If the first arc has length $\geq \pi$, we can get a new path with smaller length as in the proof of Lemma 3. Assume now that the first arc has length $< \pi$. Then γ cannot intersect the half-line from s downward vertical. Without loss of generality, we assume that t lies in the left half-plane of the vertical line ℓ through s. By Lemma 4, the last arc of γ has length at most π. Let γ' be a shortest path of type LS. If the first arc of γ is of type L, γ and γ' overlap from s up to some point p on the boundary of D_L, as in Figure 1 (b). Since two arcs of γ turn in opposite orientations, the first arc of γ is longer than that of γ'. Therefore, the subpath $\gamma'(p, t)$ is a line segment, and by Lemma 2 (i) the subpath $\gamma(p, t)$ has bigger length, which implies that γ' is shorter. If the first arc of γ is of type R, γ may intersect ℓ and let p be the last such intersection point of γ, as in Figure 1 (c). By Lemma 2 (i), $|\overline{sp}| < |\gamma(s, p)|$. Furthermore, p lies on the last arc of γ such that $\overline{sp}$ and $\gamma(pt)$ form a convex curve $\mathcal{C}$. Therefore, $\mathcal{C}$ is an outside curve of γ' and $|\gamma'| < |\mathcal{C}|$ by Lemma 2 (ii). This completes the proof that $|\gamma'| < |\gamma|$. □

We now consider the case that t lies in the interior of two unit discs tangent to s and show that any shortest path consists of two circular arcs.

Lemma 6 *Given $t \in (D_L \cup D_R)$, a path of type LSL (or RSR) can always be shortened to a path of type SL (or SR).*

Proof. Let γ be a path of type LSL consisting of an arc $L(sa)$ of D_L and a segment $\overline{ab}$ followed by another arc $L(bt)$ of a unit disc

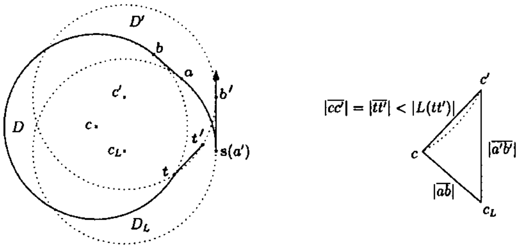

Figure 2: A path of type LSL (or RSR) can always be shortened to a path of type SL (or SR).

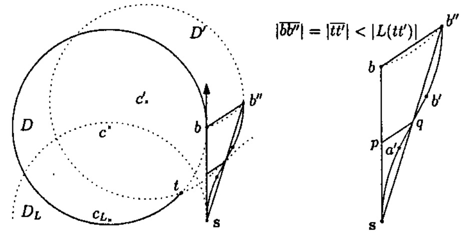

Figure 3: A path of type SL (or SR) can always be shortened to a path of type RSL (or LSR).

D, as in Figure 2. Let D' be a unit disc tangent to t and a vertical line through s at b', and let γ' be a path of type SC consisting of a line segment $\overline{sb'}$ followed by an arc $L(b't)$ of D'. Clearly, γ' is a valid path with bounded curvature.

Consider the sum of arc lengths of each path γ and γ', which is proportional the amount of changes in orientation along the path. Then γ has bigger arc length than γ' by the length of $L(tt')$ of D', where t' is the translation of t by adding the vector $\overrightarrow{cc'}$ (Note that D' is a translated copy of D by $\overrightarrow{cc'}$.) Therefore, $|\gamma| - |\gamma'| = |\overline{ab}| + |L(tt')| - |\overline{sb'}|$. Now consider the triangle connecting the centers c_L, c and c' of the three discs D_L, D and D'. Since D is a translated copy of D_L by adding the vector $\overrightarrow{c_Lc}$, $|\overline{c_Lc}| = |\overline{ab}|$. Similarly, D' is $D_L + \overrightarrow{c_Lc'}$ and $|\overline{c_Lc'}| = |\overline{sb'}|$. Since $|\overline{tt'}| < L(tt')$, by triangle inequality, $|\overline{ab}| + |L(tt')| > |\overline{sb'}|$, which shows that γ' is the shorter. The case that γ is of type RSR can be handled symmetrically. ◻

Corollary 7 *An LSL (or RSR) path can always be shortened to another path of the same type with shorter first arc.*

Proof. Given two LSL paths from s to t, let $L(sa)$ be the shorter one of two first arcs. We set s to be the configuration with location a and orientation of the counterclockwise tangent of D_L at a. Then it follows Lemma 6. ◻

Lemma 8 *Given $t \in (D_L \cup D_R)$, a path of type SL (or SR) can always be shortened to a path of type RSL (or LSR).*

Proof. Let γ be a path of type SL that starts with $\overline{sb}$ followed by an arc $L(bt)$ of disc D, and let γ' be a path of type RSL that starts with an arc $R(sa')$ of D_R followed by a line segment $a'b'$ and another arc $L(b't)$ of disc D' as in Figure 3. Let b'' be $b + \overrightarrow{cc'}$.

Then two arcs $L(bt)$ of D and $L(b''t)$ of D' differ in length by the length of $L(tt')$ of D', where t' is $t + \overrightarrow{cc'}$, as in the proof of Lemma 6. Let p be the midpoint of $\overline{sb}$, and let q be the midpoint of $\overline{a'b'}$. By Lemma 2 (ii), $|\overline{sp}| + |\overline{pq}| > |R(sa')| + |\overline{a'q}|$. If we double both sides of the inequality, it shows $|\overline{sb}| + |L(tt')| > |\overline{sb}| + |\overline{bb''}| > |R(sa')| + |\overline{a'b'}| + |L(b'b'')|$, which implies that $|\gamma'| < |\gamma|$. The case that γ is of type SR can be handled symmetrically. ◻

Corollary 9 *An RSL (or LSR) path can always be shortened to another path of the same type with longer first arc.*

Lemma 10 *Given $t \in (D_L \cup D_R)$, a path of type RS (or LS) can always be shortened to a path of type RL (or LR).*

In conclusion, a shortest path is either of type CS or type CC (LR or RL) according to the location of t, which implies Theorem 1.

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