Optimal Periodic PM Schedules Under ARI_1 Model with Different Pattern of Wear-Out Speed

Jae-Hak Lim
Department of Accounting, Hanbat National University
Taejon, 305-719, KOREA

Abstract

In this paper, we consider a periodic preventive maintenance(PM) policy in which each PM reduces the hazard rate of amount proportional to the failure intensity, which increases since the last PM and slows down the wear-out speed to that of new one. And the proportion of reduction in hazard rate decreases with the number of PMs. Our model is similar to ARI_1 proposed by Doyen and Gaudoin(2004) in the sense of reduction of hazard rate. Our model has totally different wear-out pattern of hazard rate after PM's, however, and the proportion of reduction depends on the number of PM's. Assuming that the system undergoes only minimal repairs at failures between PM's, the expected cost rate per unit time is obtained. The optimal number N of PM and the optimal period x, which minimize the expected cost rate per unit time are discussed. Explicit solutions for the optimal periodic PM are given for the Weibull distribution case.

I. Introduction

For most of the manufacturing systems in operation, it is of common practice to conduct various types of PM policies. Effective PM's of the manufacturing system are essential to minimize the system failures during the mission period and thus to maximize its productivity by preventing unexpected catastrophic failure of the system and thus by prolonging the life cycle. In such regard, the PM policy of a repairable system is of great importance in reliability theory, especially when the system exhibits deterioration in its hazard rate as it ages. Naturally, the deteriorating system may require more frequent PM's to the extent that the budget and the personnel permit. Thus, to design the most cost-effective PM schedules has been one of the most important and practical research problems in the area of system maintenance policy.

A number of PM policies have been proposed and studied in the literature. The main purpose of the PM is to slow down the degradation process of the system by taking the preventive measure while the system is still in operation. Although more frequent PM's certainly would keep the manufacturing system less likely to fail during its operation, such PM policy inevitably requires a higher cost of maintaining the system. Since Barlow and Hunter(1960) propose two types of PM policies, many authors have addressed the problem of designing the optimal schedule for the PM by determining the length of time interval between PM's to minimize the average cost rate of the system. Different types of PM policies studied in many literatures are summarized in Pham and Wang(1996) and Wang(2002).

In most of the PM policies discussed earlier, the models assume that a system undergoes PM at periodic times and is restored to as good as new after each PM. However, although the PM improves the system and slows down the degradation process, it is unlikely that the PM restores the system to the one like new for real systems. That introduces the concept of imperfect repair or imperfect PM models, which has been attracted by many researchers. Nakagawa(1980) considers an imperfect PM policy for which the system has a reduced age at each PM intervention. If the size of age reduction at each PM is equal to the PM period, then such a policy becomes a perfect PM policy. Brown and Proschan(1983) introduceimperfect repair model in which a repair restores a failed item either to the state as good as new one with probability p or to the state just prior to the failure with probability 1-p. Fontenot and Proschan(1984) develop the optimal PM schedulesbased on Brown and Proschan's(1983) model. Nakagawa(1986) proposes periodic and sequential PM policies for which the system has a different hazard functions between PM's in such a way that $h_k(t) < h_{k+1}(t)$ for any $t>0$, where $h_k(t)$ is the hazard rate in the kth period of PM. Nakagawa(1988) also considers a sequential imperfect PM policies in which the hazard rate of the system becomes $a_k h_k(t)$ after the kth PM, where a_k is an improvement factor as an increasing function of k with $a_0=1$. Taking $a_k=1$ for all $k \geq 1$, the model becomes a perfect PM model. Canfield(1986) considers a periodic PM policy for which the PM slows the degradationprocess of the system, while the hazard rate keeps monotone increase. Park, Jung and Yum(2000) derive the optimal PM schedules by associating the Canfield's PM model with various cost structures of operating the system.

Doyen and Gaudoin(2004) propose two classes of imperfect PM models based on reduction of failure intensity or virtual age, which are arithmetic reduction of intensity(ARI) model and arithmetic reduction of age(ARA) model. In ARI_1 model, which is a specific case of ARI model, PM reduces the failure intensity of amount proportional to the failure intensity, which increases since the last PM. And it is assumed that the wear-out speed is the same as before PM.

In this paper, we consider a periodic PM policy in which each PM reduces the hazard rate of amount proportional to the failure intensity, which increases since the last PM. and slows down the wear-out speed to that of new one. And the proportion of reduction in hazard rate decreases with the number of PMs. Our model is similar to ARI_1 in the sense of reduction of hazard rate. Our model has totally different wear-out pattern of hazard rate after PM's, however, and the proportion of reduction depends on the number of PM's.

The system is preventively maintained at periodic times kx and is replaced by a new system at the Nth PM, where k = 1, 2, ···, N. It is assumed that the system undergoes only minimal repair at any failure between PM's and hence, the hazard rate remains unchanged by any of minimal repairs. The expected cost rate per unit time for the proposed model is obtained. We discuss the optimal number N of the periodic PM and the optimal period x, which minimize the expected cost rate per unit time and obtain the optimal PM schedule for given cost structures of the model. Section 2 describes the periodic PM model under consideration and its assumptions. In Section 3, we derive the expression for the expected cost rate for the proposed PM policy. Section 4 presents the solutions for the optimal period and the optimal number of PM's which minimize the expected cost rate and thereby proposes the optimal PM schedule for the periodic PM policy with improvement factor. In Section 5, the optimal schedules are computed analytically the underlying failure times follow a Weibull distribution.

Notation

$h(t)$	hazard rate without PM
$h_{pm}(t)$	hazard rate with PM
x	period of PM
N	number of PM's conducted before replacement
p_i	improvement factor in hazard rate at the i-th PM
C_{mr}	cost of minimal repair at failure
C_{pm}	cost of PM
C_{re}	cost of replacement
$C(x,N)$	expected cost rate per unit time

II. Model and Assumptions

We consider a periodic PM model with an improvement factor which reduces the hazard rate of the system after PM. The followings are assumed :

(1) The system begins to operate at time $t=0$.

(2) The PM is done at periodic time kx (k = 1, 2, ⋯) where $x>0$, and is replaced by new one at the Nth PM.

(3) $h_{pm}(t)$ for $kx<t\leq(k+1)x$, which is the hazard rate after the kth PM and can be denoted by $h_{k+1}(t)$, is the sum of the reduced hazard rate due to PM at kx and the hazard rate at renewed age of (t-kx), i.e. $p_k h_{pm}(kx)+h(t-kx)$ for all $k=1,2.\cdots$, where $0\leq p_k\leq 1$..When $p_k=0$, the system after PM is restored to as good as new and when $p_k=1$, the system right after PM has the same hazard rate as that just prior to PM. For $0<p_k<1$, the hazard rate of the system is somewhat reduced after PM. After each PM, the hazard rate keeps the same degradation pattern as the new one regardless of the magnitude of p_k.

(4) $0=p_0\leq p_1\leq p_2\leq\cdots\leq p_N\leq 1$

(4) The system undergoes only minimal repair at failures between PM's.

(5) The repair and PM times are negligible.

(6) $h(t)$ is strictly increasing and convex in $t\geq 0$.

III. Expected Cost Rate Per Unit Time

In this paper, we propose a periodic PM model with an improvement factor in which the hazard rate $h(t)$ after PM k becomes $ph(t)$ when it was $h(t)$ in the period k of PM. Under this model, the hazard rate $h_{pm}(t)$ is given by

$$h_{pm}(t)=\begin{cases} h(t) & 0<t\leq x \\ h_{pm}((k-1)x+)+p_k h(x)+h(t-kx) & kx<t\leq(k+1)x \end{cases} \quad (1)$$

for $k=0,1,2,\cdots$ where $kx<t\leq(k+1)x$, $h_{pm}(0)=h(0)$ and x is the time interval between PM interventions. In the equation (1), $h_{pm}((k-1)x+)$ represent the hazard rate right after the (k-1)-th PM and $(1-p_k)h(x)$ is the amount of reduction in hazard rate due to the k-th PM.

Substituting $h_{pm}((k-1)x+)$ in the equation (1) recursively, the equation (1) can be rewritten as

$$h_{pm(t)} = \begin{cases} h(t) & \text{if } 0 < t \le x \\ \sum_{j=1}^{k} p_j h(x) + h(t-kx) & \text{if } kx < t \le (k+1)x \end{cases} \tag{2}$$

Since it is well-known from Lemma 1.1 in Fontenot and Proschan(1984) that the number of minimal repairs during the period k of PM is nonhomogeneous Poisson process(NHPP) with intensity function $\int_{kx}^{(k+1)x} h_{pm}(t)\,dt$, the expected cost rate per unit time can be obtained in the following manner:

Expected Cost Rate Per Unit Time
= [(expected cost of minimal repairs in [0, Nx))
+ (expected cost of PM in [0, Nx))
+ (expected cost of replacement)]/Nx.

Each expected cost given in the expected cost rate per unit time is obtained as follows:

(i) Expected cost of minimal repairs in [0, Nx) = $C_{mr}\left(\sum_{k=0}^{N-1}\int_{kx}^{(k+1)x} h_{pm}(t)\,dt\right)$, where $h_{pm}(t)$ is given in the equation (2).

(ii) Expected cost of PM in [0, Nx) = $(N-1)\,C_{pm}$..

(iii) Expected cost of replacement = C_{re}..

Using (i), (ii) and (iii), the expected cost rate per unit time for running the periodic PM with improvement factor during $[0,\ Nx]$ is obtained as follows:

$$\begin{aligned} C(x,N) &= \left[C_{mr}\left\{\sum_{k=0}^{N-1}\int_{kx}^{(k+1)x}\left(\sum_{j=1}^{k} p_j h(x) + h(t-kx)\right)dt\right\} + (N-1)C_{pm} + C_{re}\right]/Nx. \\ &= \left[C_{mr}\left\{xh(x)\sum_{k=0}^{N-1}\sum_{j=1}^{k} p_j + N\int_0^x h(u)\,du\right\} + (N-1)C_{pm} + C_{re}\right]/Nx. \end{aligned} \tag{3}$$

IV. Optimal Schedules for the Periodic PM Policy

In this section, we find the optimal period x^* and the optimal number N^* of PM, which minimize the expected cost rate per unit time.

We first find the optimal number of PM, when the period x is fixed. To find the optimal N*. which minimizes $C(x,N)$, we form he following inequalities.

$$C(x, N+1) \geq C(x, N)$$

and

$$C(x, N) < C(x, N-1).$$

For $0 \leq p < 1$, it can be easily shown that $C(x, N+1) \geq C(x, N)$ implies

$$xh(x)[\sum_{k=0}^{N-1}(\sum_{j=1}^{N} p_j - \sum_{j=1}^{k} p_j)] \geq \frac{C_{re} - C_{pm}}{C_{mr}}. \quad (4)$$

Similarly, the inequality $C(x, N) < C(x, N-1)$ implies

$$xh(x)[\sum_{k=0}^{N-2}(\sum_{j=1}^{N-1} p_j - \sum_{j=1}^{k} p_j)] < \frac{C_{re} - C_{pm}}{C_{mr}}. \quad (5)$$

Let $L(x, N) = xh(x)[\sum_{k=0}^{N-1}(\sum_{j=1}^{N} p_j - \sum_{j=1}^{k} p_j)]$. Then, from equations (4) and (5), we have

$$L(x, N) \geq \frac{C_{re} - C_{pm}}{C_{mr}} \quad \text{and} \quad L(x, N-1) < \frac{C_{re} - C_{pm}}{C_{mr}}. \quad (6)$$

Lemma 4.1. $L(x, N)$ is increasing in N.

proof.

$$\begin{aligned} & L(x, N+1) - L(x, N) \\ &= xh(x)\left[\sum_{k=0}^{N}\left(\sum_{j=1}^{N+1} p_j - \sum_{j=1}^{k} p_j\right) - \sum_{k=0}^{N-1}\left(\sum_{j=1}^{N+1} p_j - \sum_{j=1}^{k} p_j\right)\right] \\ &= xh(x)\left[\sum_{j=1}^{N} p_j + (N+1)p_{N+1} - \sum_{j=1}^{N} p_j\right] \\ &= xh(x)(N+1)p_{N+1} \geqq 0. \end{aligned}$$

Lemma 4.2. Suppose that $p_1 > 0$. Then $L(x, N)$ goes to infinity as N goes to infinity.

proof.

$$L(x,N) = xh(x)\left[\sum_{k=0}^{N}\left(\sum_{j=1}^{N}p_j - \sum_{j=1}^{k}p_j\right)\right] = xh(x)\left[\sum_{k=0}^{N}\sum_{j=k+1}^{N}p_j\right]$$

$$\geq xh(x)\left[\sum_{k=0}^{N}\sum_{j=k+1}^{N}p_1\right] = xh(x)p_1\frac{N(N+1)}{2}$$

which goes to infinity as N goes to infinity. ■

Theorem 4.2. Suppose that $p_1 > 0$. Then there exists a finite N^* which satisfies (6) and it is unique.

proof.

It is obvious that $L(x,N) = 0$ for $N=0$. Since from Lemma 4.1 and Lemma 4.2, $L(x,N)$ is increasing and goes to infinity as N goes to infinity, the desired result holds. ■

Next we consider the case when the number of PM, N, is fixed. To find the optimal period x^* for a given N which minimizes $C(x,N)$ in (3), we take the derivative $C(x,N)$ with respect to x and set it equal to 0. Then we have

$$x^2h'(x)\sum_{k=1}^{N-1}\sum_{j=1}^{k}p_j + Nxh(x) - N\int_0^x h(t)dt = \frac{(N-1)C_{pm} + C_{re}}{C_{mr}}. \tag{6}$$

Let $g(x)$ denotes the left-hand side of (6) and let C be the right-hand side of (6). Then

$$\frac{d}{dx}C(x,N) = 0 \Rightarrow g(x) = C, \tag{7}$$

where $g(0) = 0$ and $C > 0$.

Lemma 4.3. If $h(t)$ is strictly increasing and convex, then $g(x)$ is increasing in x.

proof.

It is easy to see that

$$\frac{d}{dx}g(x)$$
$$= 2xh'(x)\sum_{k=0}^{N-1}\sum_{j=1}^{k}p_j + x^2h''(x)\sum_{k=0}^{N-1}\sum_{j=1}^{k}p_j + Nh(x) + Nxh'(x) - Nh(x) > 0$$

iff $h(t)$ is strictly increasing and convex.

Theorem 4.4. If $h(t)$ is strictly increasing and convex function, then there exists a $x^* < \infty$ which satisfies (6) for a given integer N and it is unique.

proof.

It is also noted that $g(0)=0$. Since it is shown from Lemma 4.3 that $g(x)$ is increasing in x, it suffices that g(x) becomes ∞ as $x \to \infty$. Since $h(x)$ is increasing and convex, it is obvious that

$$xh(x) - \int_0^x h(u)du \geq \frac{xh(x)}{2}$$

which becomes ∞ as $x \to \infty$. Hence g(x) becomes ∞ as $x \to \infty$.

Thus there exists a finite and unique x^* which satisfies (6) for any giver N. ■

V. An Example

Suppose that the failure time distribution F is Weibull distribution with a scale parameter λ and a shape parameter β, is $h(t) = \beta\lambda^{\beta-1}t^{\beta-1}$ for $\beta>0$ and $t \geq 0$. As a special case, we take $\beta=2$ and $\lambda=1$ for $t \geq 0$. And also suppose that $p_k = \frac{k}{k+1}$. Then p_k increases to 1 as k goes to infinity.

Since we have $L(x, N) = xh(x)[\sum_{k=0}^{N-1}(\sum_{j=1}^{N} p_j - \sum_{j=1}^{k} p_j)] = 2x^2\left[\sum_{k=0}^{N-1}\sum_{j=k+1}^{N}\frac{j}{j+1}\right]$

$L(x, N) = xh(x)[\sum_{k=0}^{N-1}(\sum_{j=1}^{N} p_j - \sum_{j=1}^{k} p_j)] = 2x^2\left[\sum_{k=0}^{N-1}\sum_{j=k+1}^{N}\frac{j}{j+1}\right]$, the optimal number of PM's, for a given x, can be obtained by finding the smallest N which satisfies the following inequality.

$$2x^2\left[\sum_{k=0}^{N-1}\sum_{j=k+1}^{N}\frac{j}{j+1}\right] \geq \frac{C_{re} - C_{pm}}{C_{mr}} \tag{8}$$

Next suppose that N is given. Then

$$g(x) = x^2 h'(x)\sum_{k=1}^{N-1}\sum_{j=1}^{k} p_j + Nxh(x) - N\int_0^x h(t)dt = x^2\left[2\sum_{k=0}^{N-1}\sum_{j=1}^{k}\frac{j}{j+1} + N\right].$$

Hence we have

$$g(x) = C$$

$$\Rightarrow x^2\left[2\sum_{k=0}^{N-1}\sum_{j=1}^{k}\frac{j}{j+1} + N\right] = \frac{(N-1)C_{pm} + C_{re}}{C_{mr}}$$

$$\Rightarrow x^* = \left\{\left[2\sum_{k=0}^{N-1}\sum_{j=1}^{k}\frac{j}{j+1} + N\right]^{-1}\frac{(N-1)C_{pm} + C_{re}}{C_{mr}}\right\}^{\frac{1}{2}}.$$

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