Unification of lower-bound analyses of the lift-and-project rank of combinatorial optimization polyhedra

Sung-Pil Hong
Chung-Ang University

Levent Tunçel
University of Waterloo

Extended Abstract

Abstract

We present a unifying framework to establish a lower-bound on the number of semidefinite programming based, lift-and-project iterations (rank) for computing the convex hull of the feasible solutions of various combinatorial optimization problems. This framework is based on the maps which are commutative with the lift-and-project operators. Some special commutative maps were originally observed by Lovász and Schrijver, and have been used usually implicitly in the previous lower-bound analyses. In this paper, we formalize the lift-and-project commutative maps and propose a general framework for lower-bound analysis, in which we can recapture many of the previous lower-bound results on the lift-and-project ranks.

1 The Lift-and-Project Methods.

Let P be any convex subset of the d-dimensional hypercube $[0,1]^d$. P_I denotes the integral hull of P, namely the convex hull of 0-1 vectors of P. The lift-and-project methods are general procedures which take P as input and deliver P_I as output. In doing so, it is sometimes convenient to homogenize P to a cone K in $\mathbb{R}^{d+1}$ by introducing an additional coordinate which will be referred to as the 0-th coordinate.

$$K := \left\{ \lambda \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in P, \lambda \geq 0 \right\}, \tag{1}$$

$$\text{or} \quad P = \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 \\ x \end{pmatrix} \in K \right\}. \tag{2}$$

Accordingly, K_I is the homogenized cone of P_I. See Figure 1. It is clear that K is contained in

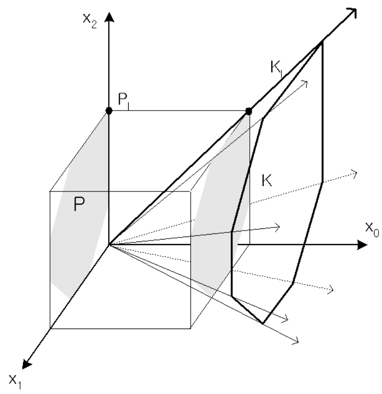

Figure 1: P, P_I, K, and K_I.

$Q \subseteq \mathbb{R}^{d+1}$, the homogenization of $[0,1]^d$. The cone Q has a very simple polyhedral structure. Denote $H_i(0) := \{x \in \mathbb{R}^{d+1} : x_i = 0\}$ and $H_i(1) := \{x \in \mathbb{R}^{d+1} : x_i = x_0\}$. Similarly, for $J \subseteq \{1,2,\dots,d\}$, write $H_J(0) := \left\{x \in \mathbb{R}^{d+1} : x_i = 0,\ i \in J\right\}$ and $H_J(1) := \left\{x \in \mathbb{R}^{d+1} : x_i = x_0,\ i \in J\right\}$. Then, for each $(d+1-k)$-dimensional face of Q, there is a set $J \subseteq \{1,2,\dots,d\}$ with $|J| = k$ and its partition $J = J_0 \cup J_1$ so that the face is given as

$$Q \cap H_{J_0}(0) \cap H_{J_1}(1). \tag{3}$$

Given a set S, its dual cone is defined as $S^* := \{x : x^T s \geq 0,\ s \in S\}$. Let L be a linear map. Then, it is easy to see that

$$y \in (LS)^* \;\Leftrightarrow\; L^T y \in S^*. \tag{4}$$

It is well known that when S is polyhedral, S^* is generated by the vectors determining the facets of S. Hence, we have

$$Q^* = \text{cone}\{e_1, \dots, e_d, f_1, \dots, f_d\}, \tag{5}$$

where e_i denotes the ith unit vector and $f_i := e_0 - e_i$. Let $K_1 \subseteq Q$ and $K_2 \subseteq Q$ be convex cones such that $K = K_1 \cap K_2$. For instance, if K is polyhedral, then K_1 and K_2 can be obtained by taking proper subsystems of the linear systems determining K. We are ready to define the lift-and-project operators N_0, N and N_+ in increasing strength. For $Y \in \mathbb{R}^{(d+1)\times(d+1)}$, consider the conditions:

$$\mathrm{diag}(Y) = Ye_0, \tag{6}$$
$$u^T Y v \geq 0,\ \forall\ u \in K_1^*,\ v \in K_2^*, \tag{7}$$

where diag : $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ maps the diagonal elements of the given matrix onto a vector. Then

$$M_0(K_1, K_2) :=$$
$$\{Y = (y_{ij})_{i,j\in\{0,1,\ldots,d\}} : Y \text{ satisfies (6), and (7)}\}.$$

Notice that (6) and (7), respectively, can be restated as follows.

$$\langle Y, f_i e_i^T \rangle$$
$$:= \mathrm{trace}\left(Y^T f_i e_i^T\right) = 0,\ \forall\ i \in \{1, 2, \ldots, d\}, \tag{8}$$
$$Y K_2^* \subseteq (K_1^*)^* = K_1. \tag{9}$$

The additional condition

$$Y \in \Sigma^{d+1},$$
$$\text{the } (d+1)\times(d+1) \text{ symmetric matrices,} \tag{10}$$

yields the stronger operator

$$M(K_1, K_2) := \{Y \in M_0(K_1, K_2) : Y \text{ satisfies (10)}\}.$$

An additional positive semidefiniteness constraint

$$Y \in \Sigma_+^{d+1}, \tag{11}$$
$$\text{the } (d+1)\times(d+1) \text{ PSD matrices,} \tag{12}$$

gives

$$M_+(K_1, K_2)$$
$$:= \{Y \in M(K_1, K_2) : Y \text{also satisfies (12)}\}. \tag{13}$$

We use $N_\sharp \in \{N_0, N, N_+\}$, and $M_\sharp \in \{M_0, M, M_+\}$, to state definitions and results for all three operators M_0, M, M_+ and N_0, N, N_+ (defined below) respectively:

$$N_\sharp(K_1, K_2) := \{Ye_0 : Y \in M_\sharp(K_1, K_2)\}. \tag{14}$$

$N_\sharp(K_1, K_2)$ is a relaxation of K_I tighter than K. We have

$$K_I \subseteq N_0(K_1, K_2) \subseteq N(K_1, K_2) \subseteq N_+(K_1, K_2) \subseteq K. \tag{15}$$

When $K_1 := K$, we can use for K_2 any convex cone such that $K \subseteq K_2 \subseteq Q$. While the choice $K_2 := K$ provides the tightest relaxations, the simplicity of Q (especially of Q^*) allows the usage of more elegant and simpler mathematical tools. Moreover, choosing $K_2 := Q$ yields a sequence of clearly tractable relaxations from a computational complexity point of view as we explain below. In this case, by (5), (9) is equivalent to

$$Ye_i, Yf_i \in K,\ i \in \{1, 2, \ldots, d\}. \tag{16}$$

For this case, we will adopt the following notation:

$$M_\sharp(K) := M_\sharp(K, Q),\ N_\sharp(K) := N_\sharp(K, Q). \tag{17}$$

Clearly, $N_\sharp$ operators can be applied iteratively:

$$K := N_\sharp^0(K),\ \ N_\sharp^t(K) := N_\sharp(N_\sharp^{t-1}(K)) \tag{18}$$
$$\text{for } t \in \{1, 2, \ldots\}.$$

2 $M_\sharp$- and $N_\sharp$-commutative maps.

Definition 1 *Suppose $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1+k}$ is a linear map. Then, L is said to be $M_\sharp$- and $N_\sharp$-commutative, respectively, if $LM_\sharp(K_1, K_2)L^T \subseteq M_\sharp(LK_1, LK_2)$ and $LN_\sharp(K_1, K_2) \subseteq N_\sharp(LK_1, LK_2)$ for every pair of closed convex cones $K_1, K_2 \subseteq Q$ (see Figure 2).*

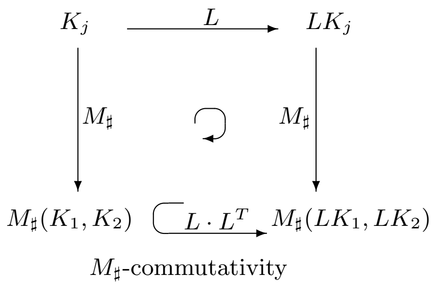

Figure 2: $M_\sharp$-commutative diagram.

Let $\tilde{e_i}$'s and $\tilde{f_i}$'s be the extreme rays of the dual cone $\tilde{Q}^*$ of the $(d+1+k)$-dimensional cone $\tilde{Q}$ spanned by the $(d+k)$-dimensional hypercube.

Theorem 2 *A linear map $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ is $M_\sharp$–commutative if and only if, for every $j \in \{1, 2, \ldots, d+1+k\}$,*

$$L^T \tilde{f}_j \tilde{e}_j^T L \in \mathrm{span}\left\{f_i e_i^T : i \in \{1, 2, \ldots, d\}\right\}. \tag{19}$$

Proof: We will prove only the sufficiency. Assume $Y \in M_\sharp(K_1, K_2)$. We first show that (19) guarantees $LYL^T \in M_\sharp(LK_1, LK_2)$. First, notice that (7), (10), and (12) are true for LYL^T regardless of (19): (10) and (12) are clearly satisfied by LYL^T. Regarding (7), due to (4), $\tilde{w} \in (LK_j)^*$ if and only if $L^T \tilde{w} \in K_j^*$ for $j \in \{1,2\}$. Therefore, $Y \in M_\sharp(K_1, K_2)$ implies $0 \le (L^T\tilde{u})^T Y (L^T \tilde{v}) = \tilde{u}^T LYL^T \tilde{v}$ for any $\tilde{u} \in (LK_1)^*$ and $\tilde{v} \in (LK_2)^*$. It remains to show (19) guarantees (6) for LYL^T.

But, by (8) the latter is equivalent to that for all $j \in \{1,2,\ldots,d+k\}$,

$$\text{trace}\left(LYL^T \tilde{f}_j \tilde{e}_j^T\right) = \text{trace}\left(Y(L^T \tilde{f}_j \tilde{e}_j^T L)\right) = 0. \tag{20}$$

From (19), $L^T \tilde{f}_j \tilde{e}_j^T L = \sum_i \lambda_i f_i e_i{}^T$ for some λ_i, $i \in \{1,2,\ldots,d\}$. Since Y satisfies (8), this implies (20).■

Corollary 3 *If, in addition, L is invertible, then the equality holds: $LM_\sharp(K_1,K_2)L^T = M_\sharp(LK_1, LK_2)$.*

Corollary 4 *If L and L' are $M_\sharp$-commutative maps, then their composite, if defined, is also $M_\sharp$-commutative.*

Lemma 5 *[5] If L is $M_\sharp$-commutative and $L^T e_0$ is parallel to e_0, then L is also $N_\sharp$-commutative.*

Corollary 6 *[5] If $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is an automorphism of Q, namely a linear map such that $LQ = Q$, then for every pair of closed convex cones $K_1, K_2 \subseteq Q$, we have $LM_\sharp(K_1,K_2)L^T = M_\sharp(LK_1,LK_2)$ and $LN_\sharp(K_1,K_2) = N_\sharp(LK_1,LK_2)$.*

A motivation of Definition 1 is that some previous lower-bound results rely on $M_\sharp$- and $N_\sharp$-commutative linear maps that are not necessarily invertible.

- *Embedding* $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ so that, for some $0 \le l \le k$,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0,1,\ldots,d\}, \\ 0 & \text{for } i \in \{d+1,\ldots,d+l\}, \\ x_0 & \text{for } i \in \{d+l+1,\ldots,d+k\}. \end{cases} \tag{21}$$

- *Duplication* $L : x \in \mathbb{R}^{d+1} \mapsto \tilde{x} \in \mathbb{R}^{d+1+k}$ so that, for a subset $\{j_1,\ldots,j_k\} \subseteq \{1,2,\ldots,d\}$,

$$\tilde{x}_i := \begin{cases} x_i & \text{for } i \in \{0,1,\ldots,d\}, \\ x_{j_{i-d}} & \text{for } i \in \{d+1,\ldots,d+k\}. \end{cases} \tag{22}$$

- *Flipping* is an automorphism that maps $e_j \mapsto f_j$, $f_j \mapsto e_j$ for each $j \in J \subseteq \{1,2,\ldots,d\}$.

In all of the above examples, one can check that for every $j \in \{1,2,\ldots,d+k\}$, there is $i \in \{1,2,\ldots,d\}$ such that

$$\left\{L^T \tilde{e}_j, L^T \tilde{f}_j\right\} = \{e_0, 0\}, \text{ or } \{e_i, f_i\}, \tag{23}$$

that is sufficient for (19). In fact, (23) describes a fairly broad class of linear maps that are both $M_\sharp$- and $N_\sharp$-commutative.

Corollary 7 *Suppose L satisfies the following conditions: 1) The first row is e_0, and 2) the rest are either, 0, e_0, e_i, or f_i for $i \in \{1,2,\ldots,d\}$. Then any positive multiple of L is both $M_\sharp$- and $N_\sharp$-commutative.*

Now, we discuss one of the key properties used in our framework for lower-bound analysis.

Lemma 8 *Let $K \subseteq \mathbb{R}^{d+1}$ and $\tilde{K} \subseteq \mathbb{R}^{d+1+k}$, respectively, be the homogenizations of the convex sets $P \subseteq [0,1]^d$ and $\tilde{P} \subseteq [0,1]^{d+k}$. Assume $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1+k}$ is an $N_\sharp$-commutative map. If L is feasible, namely $LK \subseteq \tilde{K}$, then for every $t \ge 0$, we have $LN_\sharp^t(K) \subseteq N_\sharp^t(LK) \subseteq N_\sharp^t(\tilde{K})$.*

Proof: By induction on t using the feasibility of L. ■

3 Lower Bound Analysis.

3.1 $N_\sharp$-ranks

Let Π be a 0-1 integer programming problem with the instances ι. Denote the input size of ι by $\langle\iota\rangle$ and $\Pi_n := \{\iota \in \Pi : \langle\iota\rangle \le n\}$. The rank r is a function on the quadruples $(N_\sharp, \Pi, P, n)$, where P is a initial relaxation scheme of the instances ι of Π. For each ι, let $P(\iota) \subseteq Q$ be the relaxation obtained by P applied to ι, and ℓ_ι the minimum ℓ such that $N_\sharp^\ell(P(\iota)) \subseteq P_I(\iota)$, the integral hull of $P(\iota)$. Then, the rank function r is defined as

$$r(N_\sharp, \Pi, P, n) := \max\{\ell_\iota : \iota \in \Pi_n\}. \tag{24}$$

When Π and n are clear from the context, we will simply write $r_\sharp(P) := r(N_\sharp,\Pi,P,n)$. Obviously, $r_\sharp(P)$ is a measure of efficiency of the lift-and-project methods for problem Π. However, finding an exact value of r is usually a difficult task. Therefore, the analyses are focused on finding good lower and/or upper bounds on $r_\sharp(P)$. The former is equivalent to finding an instance $\iota \in \Pi_n$, a suitable point $v(n)$ and the largest ℓ_n and such that $v(n)$ lies in the gap between $P_I(\iota)$ and $N_\sharp^{\ell_n}(P(\iota))$: $v(n) \in N_\sharp^{\ell_n}(P(\iota)) \backslash P_I(\iota)$. Then, clearly $r_\sharp(P) \ge \ell_n + 1$.

For lower-bound analysis on various combinatorial optimization problems, see also [1, 2, 3, 4].

3.2 Construction of $v(n)$

We denote by $\bar{e}$ the vector of all ones of appropriate size. Suppose $v \in \mathbb{R}_+^d$ maximizes $\bar{e}^T x$ (we assume for this discussion that the underlying combinatorial optimization problem is a maximum cardinality problem) over $N_\sharp^k(P)$. Thus, if $N_\sharp^k(P)$ is invariant under all permutations $\mathcal{S}_d$ (represented as permutation matrices), i.e.,

$$\forall R \in \mathcal{S}_d : \; x \in N_\sharp^k(P) \iff Rx \in N_\sharp^k(P).$$

Then

$$\left(\frac{1}{|\mathcal{S}_d|} \sum_{R \in \mathcal{S}_d} Rv \right) \in N_\sharp^k(P),$$

by the convexity of $N_\sharp^k(P)$. Therefore, we can assume $v = \alpha \bar{e}$ for some $\alpha \geq 0$ (we used $\bar{e}^T S v = \bar{e}^T v, \forall R \in \mathcal{S}_d$).

This kind of technique was used in previous works. It turns out, we can summarize conveniently, via $M_\sharp$-commutative maps, many of the lower-bound analyses available in the literature. They rely on a mathematical induction on the size (suitably defined) of the instances. To facilitate the presentation, we only consider the instances that are symmetric with respect to the variables. Thus, we consider essentially a unique instance of each size. Let s_k be the size of the instance at the k-th induction step. For instance, s_k can be the number of edges, nodes, or variables. Denote by P_{s_k} and K_{s_k}, respectively, the initial relaxation and its homogenization for Π_{s_k}. For simplicity, we will write $M_\sharp(s_k) := M_\sharp(K_{s_k})$ and $N_\sharp(s_k) := N_\sharp(K_{s_k})$.

3.3 Unifying approach

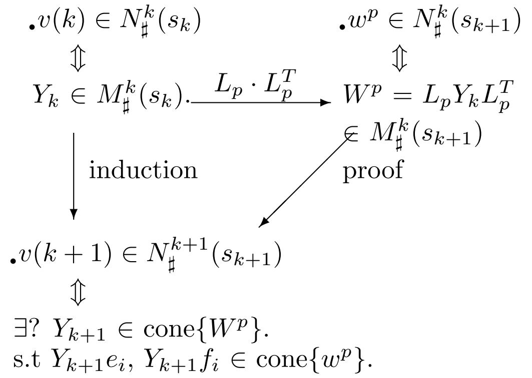

Figure 3: Unifying approach

The unifying approach focuses on constructing in a recursive manner, the sequence of proofs $Y_k \in M_\sharp^k(s_k)$ such that $v(k) = Y_k e_0$ via appropriate $M_\sharp$-commutative maps L_p. See Figure 3.

Scheme 9 *Using the symmetry of ι, P, and $v(k)$, construct $\{Y_k\}$ so that $Y_k e_0 = v(k)$, $Y_k \in M_\sharp^k(s_k)$ and $Y_{k+1} = \frac{1}{|S|} \sum_{p \in S} L_p Y_k L_p^T \in M_\sharp^{k+1}(s_{k+1})$, for some set of $M_\sharp$-commutative maps $\{L_p : p \in S\}$.*

The $M_\sharp$-commutativity of L_p's implies that $L_p Y_k L_p^T \in M_\sharp^k(s_{k+1})$ for all p. Thus, the scheme is based on the intuition that, due to the symmetry, when $L_p Y_k L_p^T \in M_\sharp^k(s_{k+1})$ for $p \in S$ then their convex combination might lie in the smaller set $M_\sharp^{k+1}(s_{k+1})$. See Figure 4.

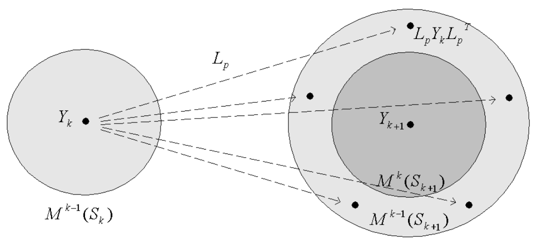

Figure 4: A schematic illustration of the proof technique.

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