Iterative 2-D/1-D Methods for the 3-D Neutron Diffusion Calculation

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1. Introduction

To remedy the problems arising from assembly homogenization and de-homogenization, several efforts have been made to solve directly the heterogeneous problem with a fine mesh and to reduce the computational burden by coupling 2-D planar with 1-D axial solutions using a transverse leakage (TL) coupling [1,2]. However, the potential for a numerical instability at a small axial mesh size has been observed [3].

Lee et al. showed that one of the two existing methods, method A, is mathematically unstable at a small mesh size while the other, method B, is always stable. They also proposed a new method for a 2-D/1-D coupling, method C, and they showed that it is always stable and it provides the best performance in terms of the convergence rate [4].

In this paper another algorithm, method D, is proposed and its stability is also investigated.

2. The New Method and Its Convergence Analysis

2.1 Existing Methods for a 2-D/1-D Coupling

2-D/1-D coupling methods begin with the axially averaged 2-D diffusion equation which can be written for each plane as in Eq. (1),

$$-\left(\frac{\partial}{\partial x}D_k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D_k\frac{\partial}{\partial y}\right)\overline{\phi}_k+\Sigma_k\overline{\phi}_k = \overline{Q}_k-\left(J_{z,k+1}-J_{z,k}\right)/h_{z,k} \quad (1)$$

and the radially averaged 1-D diffusion equation for each axial mesh which can be written as in Eq. (2),

$$-\frac{\partial}{\partial z}D_{(i,j)}\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma_{(i,j)}\phi_{(i,j)}=Q_{(i,j)} -\left(J_{x,i+1}-J_{x,i}\right)/h_x-\left(J_{y,j+1}-J_{y,j}\right)/h_y \quad (2)$$

Method A is to evaluate the TL of the 2-D equation directly from the 1-D solution [4]. This is similar to the method adopted in the DeCART code which solves the 2-D problem using MOC instead of the diffusion theory [1].

Method B is to couple the 2-D/1-D equations through the partial currents. The net currents at the top and bottom of each plane in the TL of the 2-D equation can be expressed in terms of the incoming partial currents and node average fluxes by applying a nodal method to the 1-D equation. The node average flux term in the TL can be moved to the left hand side of the equation [4]. Lee et al. used this kind of method in their work [2].

Method C is to couple the 2-D/1-D equations through a current correction factor (CCF) and the average fluxes of the lower and upper planes. The node average flux term of the given plane in the TL of the 2-D equation can be moved to the left hand side of the equation [4].

2.2 New Method for a 2-D/1-D Coupling

The new method, method D, is to use the analytic expression for the axial net currents at the interface of the planes. The axial 1-D two-node problem, Eq. (2), is solved analytically with the constraints of the average fluxes of the two nodes, the 1-D TL profiles on each node, and the continuity of the flux and the net currents at the interface. This results in an expression for the net currents at the interface of the planes in terms of the average fluxes of the two planes as follows :

$$J_{z,k}=A_k\overline{\phi}_k+B_k\overline{\phi}_{k-1}+C_k\,. \quad (3)$$

By inserting Eq. (3) into the TL of Eq. (1), the following 2-D equation is obtained.

$$-\left(\frac{\partial}{\partial x}D_k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}D_k\frac{\partial}{\partial y}\right)\overline{\phi}_k+\widetilde{\Sigma}_k\overline{\phi}_k=\overline{Q}_k+S_k\,, \quad (4)$$

where $\widetilde{\Sigma}_k=\Sigma_k-\left(A_k-B_{k+1}\right)/h_{z,k}$,

$S_k=-\left(A_{k+1}\overline{\phi}_{k+1}-B_k\overline{\phi}_{k-1}+C_{k+1}-C_k\right)/h_{z,k}$

After evaluating the right hand side of the 2-D equation, Eq. (4), it can be solved using any one of several methods. A solution to the 3-D problem is achieved by sweeping the planes until a convergence.

2.3 Model Problem for the Convergence Analysis

The model problem in reference 4 was used for the convergence analysis. The model problem is a 3-D one-group diffusion problem with a spatially flat fixed source in a homogeneous non-multiplying infinite medium. It is obvious that the solution to the problem is $\phi=Q/\Sigma$. Two basic assumptions are introduced in order to simplify the convergence analysis. These are (1) solving the 2-D problems plane by plane, which means solving them iteratively in the z-direction and (2) solving the 2-D problem by a direct inversion of the 2-D operator in a given plane. The second assumption leads to a zero radial leakage during the iterations, and simplifies Eqs (1) and (2) to:

$$\Sigma\overline{\phi}_k=Q-\left(J_{z,k+1}-J_{z,k}\right)/h\,, \quad (5)$$

$$-\frac{\partial}{\partial z}D\frac{\partial}{\partial z}\phi_{(i,j)}+\Sigma\phi_{(i,j)}=Q\,. \quad (6)$$

2.4 Convergence Analysis of the New Method

The iterative algorithm of method D applied to Eqs. (5) and (6) can be expressed by the following equation :

$$\left(\Sigma+2A/h\right)\overline{\phi}_k^{(n)}=Q+A\left(\overline{\phi}_{k-1}^{(n)}+\overline{\phi}_{k+1}^{(n-1)}\right)/h\,, \quad (7)$$

where $A = \Sigma h / \left(4\sinh^2[h/(2L)]\right)$; $L = \sqrt{D/\Sigma}$.
Inserting the first order perturbation for the average flux,

$$\bar{\phi}_k^{(n)} = Q\left(1 + \varepsilon\xi_k^{(n)}\right)/\Sigma, \quad (8)$$

into Eq. (7) yields the following equation:

$$(\Sigma + 2A/h)\xi_k^{(n)} = A\left(\xi_{k-1}^{(n)} + \xi_{k+1}^{(n-1)}\right)/h. \quad (9)$$

By inserting the following Fourier ansatz into Eq. (9)

$$\xi_k^{(n)} = a\omega^n e^{i\lambda(k+1/2)h}, \quad (10)$$

the following convergence rate can be obtained:

$$\omega = \frac{A[\cos(\tau) + i\sin(\tau)]}{\sigma + A[2 - \cos(\tau) + i\sin(\tau)]}. \quad (11)$$

the spectral radius for method D can be obtained by using the following definition :

$$\rho = \sup_{\tau\in[0,2\pi]}|\omega| . \quad (12)$$

However, the explicit form of the spectral radius is not given in this paper because it is so complicated

3. Results and Discussions

The spectral radii of the methods for the model problem are shown in Figure 1 as functions of the axial mesh size. The cross sections $D = 0.833333$ and $\Sigma = 0.02$ were used in the problem. The lines in Figure 1 are the analytic spectral radii obtained by the Fourier analysis and the dots are the numerical evaluations. As indicated, a good agreement is observed between the analytic and numerical results.

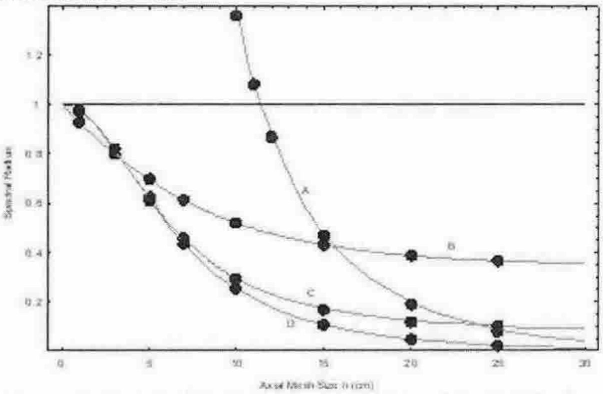

Figure 1. Spectral Radii of 2-D/1-D Coupling Methods

The spectral radius of method D is always smaller than 1, which means that method D is always stable. Method D shows the best performance among the four methods in terms of the spectral radius
It is interesting that the spectral radius of method B approaches a non-zero value while those of the others go to zero as the mesh size increases. One can show that the asymptotic spectral radii of the four methods for a large mesh size are as follows:

$$\rho_A = 4A/\Sigma h = 1/\sinh^2(h/2L), \quad (13a)$$

$$\rho_B \approx (L - 2D)^2/(L + 2D)^2, \quad (13b)$$

$$\rho_C \approx 3L^2/h^2, \quad (13c)$$

$$\rho_D \approx A/\Sigma h = 1/\left(4\sinh^2(h/2L)\right). \quad (13d)$$

The behavior of the radii for a large mesh size can be summarized as follows:

$$\rho_D < \rho_A << \rho_C << \rho_B. \quad (14)$$

While method D is the best in terms of the spectral radius, method C has several advantages over method D in terms of a practical implementation. Method C provides a greater flexibility than method D in the choice of the solution method for the 1-D solver. Only direct solution methods can be used for the 1-D solver in method D. Another drawback of method D in the multi-group case is the "fill-in"s in the 2-D operators. In a G-group case of the method D, the matrices A_k and B_k are GxG-dimensional full matrices. Therefore, the matrix $\tilde{\Sigma}_k$ in Eq. (4) is a GxG-dimensional full matrix and there are many "fill-in"s in the 2-D operator in method D. It makes the 2-D operator dense and increases the computational burden for solving the 2-D equation, Eq.(4).

4. Conclusion

In this paper a new 2-D/1-D coupling method, method D, was proposed and its convergence was analyzed. Method D provides the best performance in terms of the convergence rate. However, method C has several practical advantages over method D.

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