포화된 사질토의 동적 수치해석 기법
Variational Method for the Numerical Analysis of Saturated Granular Soils under Dynamic Loads

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SYNOPSIS : 포화토의 거동은 흙과 유체사이의 상호작용에 의해 지배된다. 특히, 동하중 작용시 포화된 사질토에서는 이러한 상호작용에 의해 급격한 액상화 현상이 유발되며 이는 동역학적, 유체역학적으로 복합적인 거동을 나타내어 복잡하므로 오직 수치해석적인 방법을 이용한 거동해석만이 가능하다. 본 연구는 사질토에서의 동적 거동해석을 위한 수치해석기법 개발에 그 목적을 두었다. 본 수치해석기법은 Hiremath (1987, 1996)의 "Dynamics for Saturated Soils" 이론에 Sandhu (1975a, 1975b, 1976)의 Variational 기법을 적용하여 개발되었으며 간극수압과 관련된 다양한 토질역학적 문제들에 적용될 수 있을 것으로 기대된다.

Keywords : Liquefaction, Saturated soils, Dynamic analysis, Numerical method, Wave propagation

1. Introduction

A review of the extent of physical and economical damage from liquefaction during recent earthquakes (Loma Prieta in 1989, Northridge in 1994, Kobe in 1995, and Turkey and Taiwan in 1999) shows that disastrous earthquakes occur on a regular basis and liquefaction of saturated granular soils is recognized as one of the major causes of ground failure during earthquakes. To address concerns with soil deformation and ground failure caused by earthquake motion, dynamics of fluid saturated porous media have been employed to analyze the liquefaction of saturated soils. Based on Biot's pioneering work (1941, 1956a, 1956b, and 1962), various theories have been proposed to explain the mechanical behavior of saturated soils under dynamic loads and methodologies have been suggested for the analysis of liquefaction. However, due to the complexity of problem in dynamics of fluid saturated soils, practical solutions are possible only through numerical approaches at the present time. Therefore, this study is focused on the development of computational procedures to produce numerical solutions incorporating dynamic theory for fluid saturated soils.

2. Theory of Dynamics of Saturated Soils

Extending Biot's theory of fluid saturated porous media, Hiremath (1987, 1996) proposed the theory of dynamics for saturated soils was proposed. This theory uses a converted coordinate system. The motion of the solid is determined with respect to a fixed reference volume which the motion of the fluid is characterized in relation to the solid. Therefore the fluid itself does not have a reference state. The definitions, relations and formulae for the use of converted coordinates in the mechanics of continua can be referred in Hiremath (1987, 1996).

2.1 Equilibrium Equations

The momentum balance equations by Hiremath (1987, 1996) in terms of the bulk stress, t_{ij} and the partial pressure, π was given as

$$\tau_{ij,j} + \rho g_i = \rho^{(1)}\ddot{u}_i + \rho^{(2)}\ddot{w}_i \tag{2.1}$$

$$\pi_{,j} + \rho^{(2)} g_i = \rho^{(2)}\ddot{u}_i^{(2)} + D\lfloor \dot{w}_i - \dot{u}_i \rfloor \tag{2.2}$$

where superposed dots denote time derivatives, u_i and w_i are the components of the displacement vectors associated with the solid and the fluid respectively, and the coefficient, D is a viscous coupling term. The density terms $\rho^{(1)}$ and $\rho^{(2)}$ are related to the density of the solid (ρ_s) and the fluid (ρ_f) respectively by means of the porosity n

$$\rho^{(1)} = (1-n)\rho_s \tag{2.3}$$

$$\rho^{(2)} = n\rho_f \tag{2.4}$$

Subtracting (2.1) from (2.2), an equilibrium equation in terms of the partial solid stress is obtained as

$$t_{ij,j}^{(1)} + \rho^{(1)} g_i = \rho^{(1)}\ddot{u}_i - D\lfloor \dot{w}_i - \dot{u}_i \rfloor \tag{2.5}$$

For small deformation, the kinematical relations are given as

$$e_{ij} = \frac{1}{2}\lfloor u_{i,j} + u_{j,i} \rfloor \tag{2.6}$$

$$\xi = w_{i,i} \tag{2.7}$$

where e_{ij} and ξ are components of the symmetric strain tensor of solid and fluid, respectively.

2.2 Constitutive relations

The constitutive equations for linear elastic fluid saturated soil are given as

$$\tau_{ij} = E_{klij} e_{kl} + \alpha M[\alpha \delta_{kl} e_{kl} + \xi]\delta_{ij} \tag{2.8}$$

$$\pi = M[\alpha e_{kl} + \xi] \tag{2.9}$$

The inverse relationships are

$$e_{kl} = C_{ijkl}(\delta_{kl} + \alpha \pi \delta_{kl})$$

$$\xi = \pi\left(\frac{1}{M} + \alpha^2 C_{ijkl}\delta_{kl}\delta_{ij}\right) - \alpha C_{ijkl}\delta_{kl}\tau_{ij} \tag{2.10}$$

Here E_{ijkl} and C_{ijkl} are components of the elasticity and compliance tensor of the elastic solid, respectively. a is the compressibility of the solid and M is that of the fluid.

2.3 Boundary and Initial Conditions

The displacement boundary conditions are

$$u_i(x,t) = \hat{u}_i(x,t) \quad \text{on} \quad S_1 \times [0,\infty)$$

$$w_i(x,t) = \hat{w}_i(x,t) \quad \text{on} \quad S_2 \times [0,\infty) \tag{2.11}$$

and the traction boundary conditions are

$$\pi_i(x,t) n_i = \hat{\pi}_i(x,t) \quad \text{on} \quad S_3 \times [0,\infty)$$

$$\tau_{ij}(x,t) n_j = T_i(x,t) = \hat{T}_i(x,t) \quad \text{on} \quad S_4 \times [0,\infty) \tag{2.12}$$

The initial conditions for the problems are

$$u(x,0) = u_0(x)$$

$$\dot{u}(x,0) = \dot{u}_0(x)$$

$$w(x,0) = w_0(x)$$

$$\dot{w}(x,0) = \dot{w}_0(x) \tag{2.13}$$

The equations (2.1) through (2.13) completely define the initial boundary value problem of small deformation of fluid saturated soil.

3. Integral Form of the Field Equation

For development of variational principles, the field equations need to be rewritten in the form of convolution product so that the time derivatives are avoided. This can be done through applying Laplace transform and taking inverse after appropriate rearrangement.

3.1 Dynamic Equilibrium

Laplace transformation of (2.1) and (2.2) followed by inversion gives

$$t*\tau_{ij,j} + F_i - \rho^{(1)} u_i - \rho^{(2)} w_i = 0 \tag{3.1}$$

$$t*\pi_{,j} + G_i - \rho^{(2)} w_i - D[w_i - u_i] = 0 \tag{3.2}$$

where

$$F_i = t*\rho b_i + \rho^{(1)}[u_i(0) - t\cdot\dot{u}_i(0)] + \rho^{(2)}[w_i(0) - t\cdot\dot{w}_i(0)] \tag{3.3}$$

$$G_j = t*\rho^{(2)} b_i + \rho^{(2)}[w_i(0) + t\cdot\dot{w}_i(0)] + D[t\cdot w_i(0) - t\cdot u_i(0)] \tag{3.4}$$

The symbol "*" denotes the convolution product defined as

$$f*g = \int_0^t f(\tau) g(t-\tau) d\tau \tag{3.5}$$

3.2 Constitutive Equations

Equations (2.8) to (2.10) must be restated so that the constitutive relations show the dependence of quantities appearing in the equilibrium equations upon corresponding kinematical quantities in them.

$$\begin{aligned} t*\tau_{ij} &= t*E_{ijkl} e_{kl} + t*\alpha M \delta_{ij}(\alpha\ \delta_{kl} e_{kl} + \xi) \\ t*\pi &= t*M(\alpha \delta_{ij} e_{ij} + \xi) \\ t*e_{ij} &= t*C_{ijkl}(\tau_{kl} - \alpha\pi\delta_{kl}) \\ t*\xi &= t*\pi\left(\frac{1}{M} + \alpha^2 C_{ijkl}\delta_{kl}\delta_{ij}\right) - t*\alpha C_{ijkl}\delta_{kl}\tau_{ij} \end{aligned} \tag{3.6}$$

4. Variational Principles for Dynamics of Fluid Saturated Soils

4.1 Field Equations

The integral form of field equations (3.1) through (3.6) is can be written in a self-adjoint matrix form;

$$A(u) = f \text{ on } R\times[0,\infty) \tag{4.1}$$

Here,

$$A = \begin{bmatrix} \rho & \rho_2 & 0 & -L & 0 & 0 \\ \rho_2 & \rho_2/f + 1*(1/k) & -t*\dfrac{\partial}{\partial m} & 0 & 0 & 0 \\ 0 & t*\dfrac{\partial}{\partial m} & 0 & 0 & 0 & -t* \\ L & 0 & 0 & 0 & -t* & 0 \\ 0 & 0 & 0 & -t* & P & t*\alpha M\delta_{ij} \\ 0 & 0 & -t* & 0 & t*\alpha M\delta_{kl} & t*M \end{bmatrix} \tag{4.2}$$

where

$$L = \left(\frac{1}{2}\right)t*\left(\delta_{lm}\frac{\partial}{\partial k} + \delta_{km}\frac{\partial}{\partial l}\right) \tag{4.3}$$

$$P = t*(E_{ijkl} + \alpha^2 M\delta_{ij}\delta_{kl}) \tag{4.4}$$

$$u = \begin{bmatrix} u_m \\ w_m \\ \pi \\ \tau_{ij} \\ e_{kl} \\ \xi \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} F_m \\ G_m \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.5}$$

Elements of matrix, A satisfy self-adjointness. The operators on the diagonal are symmetric and the off-diagonal operators constitute adjoint pairs with respect to the bilinear mapping. Consistent boundary conditions for the equations (4.1) are

$$\begin{aligned} -t*u_i n_j &= -t*\hat{u}_i\, n_j && \text{on} \quad S_1\times[0,\infty) \\ -t*w_i n_i &= -t*\hat{w}_i\, n_i && \text{on} \quad S_2\times[0,\infty) \\ t*\pi n_i &= t*\hat{\pi}\;\, n_i && \text{on} \quad S_3\times[0,\infty) \\ t*\tau_{ij} n_j &= t*\hat{T}_i && \text{on} \quad S_4\times[0,\infty) \end{aligned} \tag{4.6}$$

Consistent form of the internal jump discontinuities is

$$-t*(u_i n_j)' = -t*(g_1)_i n_j \quad \text{on} \quad S_{1i}\times[0,\infty)$$
$$-t*(w_i n_i)' = -t*g_2 \quad \text{on} \quad S_{2i}\times[0,\infty)$$
$$t*(\pi n_i)' = t*g_3 n_i \quad \text{on} \quad S_{3i}\times[0,\infty)$$
$$t*(\tau_{ij} n_j)' = t*g_4 n_i \quad \text{on} \quad S_{4i}\times[0,\infty) \tag{4.7}$$

Here, surface S_{1i}, S_{2i}, S_{3i} and S_{4i} are embedded in the interior of R. Operators in the self-adjoint operator matrix equation (4.1) have the following relationships;

$$\langle t*u_{i,j}, \tau_{ij}\rangle_R = -\langle t*u_i, \tau_{ij,j}\rangle_R + \langle t*u_i n_j, \tau_{ij}\rangle_{S_1} + \langle t*u_i, \tau_{ij} n_j\rangle_{S_4}$$
$$+\langle t*(u_i n_j)', \tau_{ij}\rangle_{S_{1i}} + \langle t*u_i, (\tau_{ij} n_j)'\rangle_{S_{4i}} \tag{4.8}$$
$$\langle t*w_i, \pi_{,i}\rangle_R = -\langle t*w_{i,i}, \pi\rangle_R + \langle t*w_i n_i, \pi\rangle_{S_2} + \langle t*w_i, \pi n_i\rangle_{S_3}$$
$$+\langle t*(w_i n_i)', \pi\rangle_{S_{2i}} + \langle t*w_i, (\pi n_i)'\rangle_{S_{3i}} \tag{4.9}$$

In (4.8) and (4.9), the $\langle \ , \ \rangle_R$ can be evaluated as the sum of quantities evaluated over subregions of R such that all the surfaces $S_{1i}, S_{2i}, S_{3i}, S_{4i}$ are contained in the union of the boundaries of these subregions.

4.2 A General Variational Principle

For the operator equation (4.1), the governing function following (A.16) is defined as;

$$\Omega(u) = \langle \rho u_i, u_i\rangle_R + 2\langle \rho_2 w_i, u_i\rangle_R - \langle t*\tau_{ij,j}, u_i\rangle_R + \left\langle \left(\frac{\rho_2}{f} + 1*\frac{1}{k}\right) w_i, w_i \right\rangle_R$$
$$-\langle t*\pi_{,i}, w_i\rangle_R + \langle t*w_{i,i}, \pi\rangle_R - 2\langle t*\xi, \pi\rangle_R + \langle t*u_{i,j}, \tau_{ij}\rangle_R$$
$$-2\langle t*e_{ij}, \tau_{ij}\rangle_R + \langle t*(E_{ijkl} + \alpha^2 M\delta_{ij}\delta_{kl})e_{kl}, e_{ij}\rangle_R + 2\langle t*\alpha M\delta_{ij}e_{ij}, \xi\rangle_R + \langle t*M\xi, \xi\rangle_R$$
$$-2\langle u_i, F_i\rangle_R - 2\langle w_i, G_i\rangle_R - \langle \tau_{ij}, t*(u_i - 2u_i)n_j\rangle_{S_1} - \langle \pi, t*(w_i - 2w_i)n_i\rangle_{S_2}$$
$$+\langle w_i, t*(\pi - 2\pi)n_i\rangle_{S_3} + \langle u_i, t*(\tau_{ij}n_j - 2T_i)\rangle_{S_4} - \langle \tau_{ij}, t*((u_i n_j)' - 2(g_1)_i n_j)\rangle_{S_{1i}}$$
$$-\langle \pi, t*((w_i n_i) - 2g_2)\rangle_{S_{2i}} + \langle w_i, t*((\pi n_i)' - 2g_3 n_i)\rangle_{S_{3i}} + \langle u_i, t*((\tau_{ij} n_j)' - 2g_4 n_i)\rangle_{S_{4i}} \tag{4.10}$$

The Gateaux differential of this function along $v=\{\overline{u}_i,\overline{w}_i,\overline{\pi},\overline{\tau}_{ij},\overline{e}_{ij},\overline{\xi}\}$ is;

$$\Delta_V\Omega(u)=\left\langle\overline{u}_i,\rho u_i+\rho_2 w_i-t*\tau_{ij,j}-2F_i\right\rangle_R+\left\langle u_i,\rho\overline{u}_i+\rho_2\overline{w}_i-t*\overline{\tau}_{ij,j}\right\rangle_R$$

$$+\left\langle\overline{w}_i,\rho_2 u_i+(\frac{\rho_2}{f}+1*\frac{1}{k})w_i-t*\pi_{,i}-2G_i\right\rangle_R+\left\langle w_i,\rho_2\overline{u}_i+(\frac{\rho_2}{f}+1*\frac{1}{k})\overline{w}_i-t*\overline{\pi}_{,i}\right\rangle_R$$

$$-\left\langle\overline{\pi},t*w_{i,i}-t*\xi\right\rangle_R+\left\langle\pi,t*\overline{w}_{i,i}-t*\overline{\xi}\right\rangle_R-\left\langle\overline{\tau}_{ij},t*u_{i,i}-t*e_{i,j}\right\rangle_R+\left\langle\tau_{ij},t*\overline{u}_{i,i}-t*\overline{e}_{ij}\right\rangle_R$$

$$-\left\langle\overline{e}_{ij},-t*\tau_{ij}+t*(E_{ijkl}+\alpha^2M\delta_{ij}\delta_{kl})e_{kl}+\alpha M\delta_{ij}\xi\right\rangle_R$$

$$-\left\langle e_{ij},-t*\overline{\tau}_{ij}+t*(E_{ijkl}+\alpha^2M\delta_{ij}\delta_{kl})\overline{e}_{kl}+\alpha M\delta_{ij}\overline{\xi}\right\rangle_R$$

$$+\left\langle\overline{\xi},-t*\pi+t*\alpha M\delta_{kl}e_{kl}+t*M\xi\right\rangle_R+\left\langle\xi,-t*\overline{\pi}+t*\alpha M\delta_{kl}\overline{e}_{kl}+t*M\overline{\xi}\right\rangle_R$$

$$-\left\langle\overline{\tau}_{ij},t*(u_in_j-2\hat{u}_in_j)\right\rangle_{S_1}-\left\langle\tau_{ij},t*\overline{u}_in_j\right\rangle_{S_1}$$

$$-\left\langle\overline{\pi},t*(w_in_i-2\hat{w}_in_i)\right\rangle_{S_2}-\left\langle\pi,t*\overline{w}_in_i\right\rangle_{S_2}$$

$$-\left\langle\overline{w}_i,t*(\pi n_i-2\hat{\pi}_in_i)\right\rangle_{S_3}-\left\langle w_i,t*\overline{\pi}n_i\right\rangle_{S_3}$$

$$-\left\langle\overline{u}_i,t*(\tau_{ij}n_j-2\hat{T}_i)\right\rangle_{S_4}-\left\langle u_i,t*\tau_{ij}n_j\right\rangle_{S_4}$$

$$-\left\langle\overline{\tau}_{ij},t*(u_in_j)'-2(g_1)_in_j\right\rangle_{S_{1i}}-\left\langle\tau_{ij},t*(\overline{u}_in_j)'\right\rangle_{S_{1i}}$$

$$-\left\langle\overline{\pi},t*((w_in_i)'-2(g_2)\right\rangle_{S_{2i}}-\left\langle\pi,t*(\overline{w}_in_i)\right\rangle_{S_{2i}}+\left\langle\overline{w}_i,t*(\pi n_i)'-2g_3n_i)\right\rangle_{S_{3i}}+\left\langle w_i,t*(\overline{\pi}n_i)'\right\rangle_{S_{3i}}$$

$$-\left\langle\overline{u}_i,t*((\tau_{ij}n_j)'-2(g_4)n_i\right\rangle_{S_{4i}}+\left\langle u_i,t*(\overline{\tau}_{ij}n_j)'\right\rangle_{S_{4i}} \quad (4.11)$$

Using equation (4.8) and (4.9), the gateaux differential can be rewritten as;

$$\Delta_V\Omega(u)=2\left\langle\overline{u}_i,\rho u_i+\rho_2 w_i-t*\tau_{ij,j}-F_i\right\rangle_R-2\left\langle\overline{\pi},t*w_{i,i}-t*\xi\right\rangle_R+2\left\langle\tau_{ij},t*u_{i,j}-t*e_{ij}\right\rangle_R$$

$$-2\left\langle\overline{\pi},t*w_{i,i}-t*\xi\right\rangle_R+2\left\langle\tau_{ij},t*u_{i,j}-t*e_{ij}\right\rangle_R-2\left\langle\overline{e}_{ij},-t*\tau_{ij}+t*(E_{ijkl}+\alpha^2M\delta_{ij}\delta_{kl})e_{kl}+\alpha M\delta_{ij}\xi\right\rangle_R$$

$$+2\left\langle\overline{\xi},-t*\pi+t*\alpha M\delta_{kl}e_{kl}+t*M\xi\right\rangle_R-2\left\langle\overline{\tau}_{ij},t*(u_in_j-u_in_j)\right\rangle_{S_1}$$

$$-2\left\langle\overline{\pi},t*(w_in_i-w_in_i)\right\rangle_{S_2}-2\left\langle\overline{w}_i,t*(\pi n_i-\pi_in_i)\right\rangle_{S_3}-2\left\langle\overline{u}_i,t*(\tau_{ij}n_j-T_i)\right\rangle_{S_4}$$

$$-2\left\langle\overline{\tau}_{ij},t*(u_in_j)'-(g_1)_in_j\right\rangle_{S_{1i}}-2\left\langle\overline{\pi},t*((w_in_i)'-(g_2)\right\rangle_{S_{2i}}+2\left\langle\overline{w}_i,t*((\pi n_i)'-g_3n_i)\right\rangle_{S_{3i}}$$

$$+2\left\langle\overline{u}_i,t*((\tau_{ij}n_j)'-(g_4)n_i\right\rangle_{S_{4i}} \quad (4.12)$$

The Gateaux differential vanishes if and only if all the field equations along with the boundary conditions and the jump conditions are satisfied because of linearity and non-degeneracy of bilinear mapping. Hence, vanishing of $\Delta_V\Omega(u)$ for all $v \in w$ implies satisfaction of (4.1), (4.7), and (4.8).

5. Summary

A numerical procedure to produce computational solutions incorporating Hiremath's dynamic theory for saturated soils is developed. In the theory, the motion of the solid is described with respect to its reference configuration but the motion of the fluid is described as relative to the solid. To transform the coupled initial-boundary value problem of wave equations into an equivalent variational problem, the field equations are re-written in the form of convolution product so that the time derivatives are avoided. The set of fluid variables are regarded as a multiple in the admissible space whose elements are defined in the spatial region. A solution of the mixed problem is an admissible state of the field variables, which satisfies the filed equation, the initial conditions and the boundary conditions to the problem. Extensions of this study are applicable for analyzing multi-phase systems such as coupled problems with the simultaneous presence of water and air in which air pressure plays important role or that of water and oil for the treatment of oil reservoirs. The extensions can be done with the allowing the field equations of motion to contain two different fluids.

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APPENDIX

MATHEMATICS

A.1 Boundary Value Problem

The linear vector space W consisting of all admissible states is referred to as the product space

$$W = W_1 \times W_2 \times \cdots\cdots\cdots\cdots\cdots\cdots \times W_n \qquad \text{(A.1)}$$

where W_i is an subspace whose elements represent the admissible state for an specific field variable, *u*. Consider the boundary vale problem given as

$$A(u) = f \text{ on } R \text{ x } [0,\infty)$$

$$C(u) = g \text{ on } \partial R \text{ x } [0,\infty) \qquad \text{(A.2)}$$

where *R* is an open connected region of interest, ∂R is the boundary of *R*, and A, C are linear operator matrices. The field operator *A* and the boundary operator C are bounded and defined such that

$$A : W_R \rightarrow V_R$$

$$C : W_{\partial R} \rightarrow V_{\partial R} \qquad \text{(A.3)}$$

V_R, $V_{\partial R}$ are linear vector spaces defined on the regions indicated by the subscripts and W_R, $W_{\partial R}$ are subsets in V_R, $V_{\partial R}$, respectively. Throughout, *A* and C are assumed to be linear so that

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \quad u, v \in W_R$$

$$C(\alpha u + \beta v) = \alpha C(u) + \beta C(v) \quad u, v \in W_{\partial R} \qquad \text{(A.4)}$$

where α, β are arbitrary scalars. Solution of boundary value problem implies determination of $u \in W_R$ for given $f \in V_R$ and $g \in V_{\partial R}$ subject to the satisfaction of equation (A.1) – (A.4).

A.2 Bilinear Mapping

A bilinear mapping $B : W \times V \rightarrow S$, where W, V, S are linear vector spaces, for given $w \in W, v \in V,$ is defined as a function to assign an element in S corresponding to an ordered pair (w, v). B is said to be bilinear if

$$B(\alpha w_1 + \beta w_2, v) = \alpha B(w_1, v) + \beta B(w_2, v) \qquad \text{(A.5)}$$

$$B(w, \alpha v_1 + \beta v_2) = \alpha B(w, v_1) + \beta B(w, v_2) \qquad \text{(A.6)}$$

where α, β are scalars. The notation can be used as

$$B_R(w, v) = \langle w, v \rangle_R \qquad \text{(A.7)}$$

B is said to be non-degenerate if

$$\langle w, v \rangle_R = 0 \quad w \in W \text{ if and only if } v = 0 \qquad \text{(A.8)}$$

For $W = V$, B is symmetric if

$$\langle w, v \rangle_R = \langle v, w \rangle_R \tag{A.9}$$

A.3 Self-Adjoint Operator

Let $A: V \to W$ be an operator on the linear vector space V defined on spatial region R. Operator A^* is said to be adjoint of A with respect to a bilinear mapping $\langle \, , \, \rangle_R : W \times W \to S$ if

$$\langle w, Av \rangle_R = \langle v, A^* w \rangle_R + D_{\partial R}(v, w) \tag{A.10}$$

for all $w \in W$ and $v \in V$. Here, $D_{\partial R}(v, w)$ represents quantities associated with the boundary ∂R of R. If $A = A^*$, then A is said to be self-adjoint. In particular, a self-adjoint operator A on V is symmetric with respect to the bilinear mapping if $V = W$ and

$$\langle w, Av \rangle_R = \langle v, Aw \rangle_R \tag{A.11}$$

A. 4 Gateaux Differential of a Function

Considering V and S as linear vector spaces, the Gateaux differential of a continuous function $F: V \to S$ is defined as

$$\Delta_V F(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[F(u + \lambda v) - F(u) \right] \tag{A.12}$$

provided the limit exists. v is referred to as the 'path' and λ is a scalar. For $u, v \in V$, $u + \lambda v \in V$. Equation (A.12) can be equivalently written as

$$\Delta_V F(u) = \frac{d}{d\lambda} F(u + \lambda v) \Big|_{\lambda = 0} \tag{A.13}$$

A.5 Basic Variational Principles

For the boundary value problem given by (A.1) with homogeneous boundary condition, Mikhlin (1965) showed the functional, $\Omega(u)$ to be a minimum value for the unique solution u_0 with self-adjoint, positive definite operator A,

$$\Omega(u) = \langle Au, u \rangle_R - 2\langle u, f \rangle_R \tag{A.14}$$

where $\langle \, , \, \rangle_R$ denotes inner product over the separable space of square functions. The u_0 that minimizes the functional (A.14) is the solution of the problem (A.1). Taking Gateaux differential of (A.14),

$$\begin{aligned} \Delta_V \Omega(u) &= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\langle A(u + \lambda v, u + \lambda v \rangle - 2\langle u + \lambda v, f \rangle - \langle Au, u \rangle + 2\langle u, f \rangle \right] \\ &= \langle Au, v \rangle + \langle Av, u \rangle - 2\langle v, f \rangle \\ &= 2\langle v, Au - f \rangle = 0 \end{aligned} \tag{A.15}$$

In (A.15), the linearity and self-adjointness of A with respect to the bilinear mapping and the symmetry of the bilinear mapping are assumed. The Gateaux differential vanishes at the solution u_0 such that $Au_0 - f = 0$. For the vanishing of the Gateaux differential at $u = u_0$ to imply $Au_0 - f = 0$, the bilinear mapping has to be into the real line and the operator must be positive. However, in general, it is only necessary to use vanishing of the Gateaux differential as equivalent to (A.1) being satisfied. The governing function for the operator equation (A.2) can be defined as

$$\Omega = \left\langle u_i, A_{ij}u_j - 2f_i \right\rangle_R + \left\langle u_i, C_{ij}u_j - 2g_i \right\rangle_{\partial R} \tag{A.16}$$

A.6 Consistent Boundary and Initial Discontinuity

Sandhu (1976) pointed out that appropriate boundary terms should be included in the governing function even if they are homogeneous. This is important for approximation procedures such as the finite element method, where the functions of limited smoothness are used. The boundary operators must be in a form consistent with the field operator. Considering the boundary value problem of multi-variables given (A.5) and (A.6), Sandhu (1976) defined consistency of boundary operators with the field operators to be the property;

$$D_{\partial R}(u_i, u_j) = \left\langle v_j, \sum_j^n C_{ij}u_j \right\rangle_{\partial R} - \sum_j^n \left\langle u_j, C_{ij}v_i \right\rangle_{\partial R}, \quad i = 1,2,\cdots\cdots\cdots,n \tag{A.17}$$

To find an approximation to the exact solution by the finite element method, the function space with limited smoothness over the entire domain is sometimes used. In order to properly handle this limited smoothness problem in the variational formulation, Sandhu (1976) introduced internal discontinuity conditions in the form;

$$C(u)' = g \text{ on } \partial R_i \tag{A.18}$$

where a prime denotes the internal jump discontinuity along element boundary ∂R_i embedded in the region R. Sandhu and salaam (1975a) and Sandhu (1975b) showed that this condition can be included explicitly in the governing function.