A New Method of Collision Mode Evolution for Three-Dimensional Rigid Body Impact With Friction

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Abstract: In presence of collision between two rigid bodies, they exhibit impulsive behavior to generate physically feasible state. When the frictional impulse is involved, collision resolution can not be easily made based on a simple Newton's law or Poisson's law, mainly due to possible change of collision mode during collision, For example, sliding may change to sticking, and then sliding resumes. We first examine two conventional methods: the method of mode evolution by differential equation, and the other by linear complementarity programming. Then, we propose a new method for mode evolution by solving only algebraic equations defining mode changes. Further, our method attains the original nonlinear impulse cone constraint. The numerical simulation will elucidate the advantage of the proposed method as an alternative to conventional ones.

Keywords: Rigid body collision, friction, Poisson's law

1. Introduction

Consider two bodies colliding at a single point at time t. Collision is defined as a contact with a penetrating normal velocity. The relative velocity between two contact points is denoted by $v(t) = (v_x(t), v_y(t), v_z(t))^T$ where $v_z(t)$ is the normal component and $(v_x(t), v_y(t))$ are tangential ones. For virtually instantaneous collision duration, they experience impulsive momentum change entailing sign reversal of the normal velocity. A concise approximation of momentum change, during a pre-impact instant t^- to a post-impact instant t^+, is expressed by the impulse-momentum equation written as

$$v(t^+) - v(t^-) = H\Gamma(t^- : t^+), \tag{1}$$

or in a componentwise form as

$$\begin{bmatrix} v_x(t^+) - v_x(t^-) \\ v_y(t^+) - v_y(t^-) \\ v_z(t^+) - v_z(t^-) \end{bmatrix} = \begin{bmatrix} X & U & W \\ U & Y & V \\ W & V & Z \end{bmatrix} \begin{bmatrix} \Gamma_x(t^- : t^+) \\ \Gamma_y(t^- : t^+) \\ \Gamma_z(t^- : t^+) \end{bmatrix}, \tag{2}$$

where $\Gamma_k(t^- : t^+)$ is the impulse defined by

$$\Gamma_k(t^- : t^+) = \int_{t^-}^{t^+} f_k(\tau) d\tau, \quad k = x, y, z, \tag{3}$$

and H is the inverse matrix of the effective inertia at the contact point. Given a contact point location and the configurations of two bodies, it can be computed. Every force component of nonimpulsive nature vanishes in integrating from t^- to t^+ as the duration is very infinitesimal. Only the impulsive forces remain.

Given the velocity at t^-, called the pre-impact velocity, one has to find six unknowns consisting of the post-impact velocity vector, i.e. $v(t^+)$, and the impulse vector during the collision, i.e. $\Gamma(t^- : t^+)$. Taking into account that the impulse-momentum equation provides three equations, three additional equations should be provided to determine the unknowns. This process is referred to as the *collision resolution*. Hereinafter, the impulse-momentum equation for a duration $[\tau_1, \tau_2]$ is denoted by $IM(\tau_1 : \tau_2)$, and each component as $IM_x(\tau_1 : \tau_2)$, $IM_y(\tau_1 : \tau_2)$, and $IM_z(\tau_1 : \tau_2)$.

When collision occurs without friction, tangential impulse components become zero, providing two equations such as $\Gamma_x(t^- : t^+) = \Gamma_y(t^- : t^+) = 0$. Classically, one additional constitutive equation is given by an impulse law. There are two such laws: Newton's and Poisson's law. Newton's law is based on a kinematic coefficient of restitution ϵ such that

$$v_z(t^+) = -\epsilon v_z(t^-). \tag{4}$$

One can partition the collision duration by two phases: the *compression* phase accumulating impulse due to compression, and the *expansion* transferring the impulse to momentum change. Poisson's law relates the impulses during two phases by

$$\Gamma_z^+ = \epsilon \Gamma_z^- \tag{5}$$

where $\Gamma_z^+ = \Gamma_z(t : t^+)$ and $\Gamma_z^- = \Gamma_z(t^- : t)$. It is assumed that the end of compression and the beginning of expansion occurs at instant t. The coefficient ϵ is now called the kinetic coefficient of restitution.

Two laws lead to an identical collision resolution for frictionless collision. When sliding velocity changes direction during collision, Newton's law based collision resolution cannot account properly for energy loss due to friction. This sometimes results in energy gain during collision [1], which is definitely unacceptable. Hence, for frictional collision, two laws produce distinctive results.

2. Collision Resolution due to a Single Collision

It is assumed that $v_z \leq 0$ (with respect to the our contact coordinate frame) is only feasible due to nonpenetration. Similarly, only the negative normal force and impulse can be generated.

2.1. Mode evolution by differential equation

During sliding, frictional force is constrained to lie on the boundary of the friction cone

$$f_x = \frac{\mu_d v_x}{\sqrt{v_x^2 + v_y^2}} f_z; \quad f_y = \frac{\mu_d v_y}{\sqrt{v_x^2 + v_y^2}} f_z$$

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with the opposite direction to the sliding velocity, where μ_d is the dynamic friction coefficient. It is transferred to the impulse domain as

$$\frac{d\Gamma_x}{d\Gamma_z} = \frac{\mu_d v_x}{\sqrt{v_x^2 + v_y^2}}; \quad \frac{d\Gamma_y}{d\Gamma_z} = \frac{\mu_d v_y}{\sqrt{v_x^2 + v_y^2}}, \tag{6a}$$

since $d\Gamma_k = \dfrac{d\Gamma_k(t^- : \tau)}{d\tau} d\tau = f_k(\tau) d\tau$ for $k = x, y, z$.
Differentiating the impulse-momentum equation $IM(t^- : \tau)$ with respect to the normal impulse $\Gamma_z(t^- : \tau)$ yields

$$\frac{d}{d\Gamma_z} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = H\xi(v_x, v_y), \quad \xi(v_x, v_y) = \begin{bmatrix} \dfrac{\mu_d v_x}{\sqrt{v_x^2 + v_y^2}} \\ \dfrac{\mu_d v_y}{\sqrt{v_x^2 + v_y^2}} \\ 1 \end{bmatrix}, \tag{7}$$

where $\xi(v_x, v_y)$ is the sliding direction vector. The rows of the matrix H are denoted by H_x, H_y, and H_z, respectively. This is an ordinary differential equation with respect to Γ_z. It is called the sliding differential equation [2].
Keller [3] reached the sliding differential equation by time-scaling. In planar case, Routh's algebraic incremental model is equivalent to analytically solving the sliding differential equation, as shown by Wang and Mason [4].
For three-dimensional case, the sliding differential equation is first integrated with respect to Γ_z from 0 while checking whether the normal velocity v_z becomes zero (signaling the end of compression) and whether the sliding velocity becomes zero (indicating the occurrence of sticking), i.e. $\sqrt{v_x^2 + v_y^2} = 0$. If the end of compression is signaled, the value of Γ_z at the instant is latched to Γ_z^-. Then, the sliding differential equation is integrated further from Γ_z^- to $(1+\epsilon)\Gamma_z^-$, while checking for sticking. The termination condition reflects Poisson's hypothesis (5). If no such event arises, one gets finally the post-impact velocity $v(t^+)$.
If sticking occurs during either compression or expansion phase, then its stability is examined, which determines whether the sticking would prevail throughout the remaining phases or sliding would resume. In either case, the remaining periods can be integrated analytically using constancy of direction of the sliding vector [2].
Mirtich [2] proposed to use the sliding differential equation with different parameters instead of Γ_z. The normal velocity v_z is used as an independent parameter during compression, while the normal work w_z during expansion. This enables to explicit specification of the terminal condition. Of particular interest is that the Stronge's energetic coefficient of restitution [5] can be applied to stipulate explicitly the energy loss during collision process.

2.2. Mode evolution by linear complementarity programming (LCP)

One can impose impulse cone constraint

$$\Gamma_x^2 + \Gamma_y^2 \le \mu^2 \Gamma_z^2 \tag{8}$$

instead of friction cone. This impulse cone constraint is assumed to hold for the compression and expansion phases, that is each Γ_k can be either $\Gamma_k^- = \Gamma_k(t^- : t)$ or $\Gamma_k^+ = \Gamma_k(t : t^+)$ for $k = x, y, z$.

Glocker and Pfeiffer [6] formulated two linear complementarity conditions for each phase in case of *planar* collisions. The compression is subject to

$$-v_z(t) \ge 0 \quad \perp \quad -\Gamma_z^- \ge 0 \tag{9}$$

at the end of compression. The symbol '⊥' denotes that two variables are complementary, i.e. $x \ge 0 \perp y \ge 0$ for two vectors of same dimension means $x_i \cdot y_i = 0$ for all i. The expansion is defined by $\Gamma_z^+ = \epsilon\Gamma_z^- + \widetilde{\Gamma}_z$, where the additional impulse $\widetilde{\Gamma}_z$ is complementary to the post-impact normal velocity $v_z^+ = v_z(t^+)$, i.e.

$$-v_z^+ \ge 0 \quad \perp \quad -\widetilde{\Gamma}_z \ge 0. \tag{10}$$

According to their method, the normal impulse during expansion phase is composed of two components, one transferred from (the fraction of) the normal impulse during compression phase, i.e. Γ_z^-, and the other required to prevent interpenetration at the end of expansion.
The same model was adopted in Anitescu and Potra [7], while developing three-dimensional collision resolution method. Approximating the quadratic impulse cone constraint using an inscribed polyhedral cone having p edges [8], [7]

$$\widehat{IC} = \{\hat{z}\Gamma_z + \chi\Gamma_t | -\Gamma_z \ge 0, \Gamma_t \ge 0, e^T\Gamma_t \le -\mu\Gamma_z\}, \tag{11}$$

where $\hat{z} = (0,0,1)^T$ and $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^p$. The vector Γ_t consists of $(\Gamma_t^{(1)}, \Gamma_t^{(2)}, \cdots, \Gamma_t^{(p)})^T \in \mathbb{R}^p$. The matrix χ consists of p columns of the form $\begin{bmatrix} \chi_x^{(j)} & \chi_y^{(j)} & 0 \end{bmatrix}^T$. The columns of the matrix χ span the tangent plane at the contact. It is assumed that for every i there is a j such that $\chi^{(i)} = -\chi^{(j)}$. One can simply set

$$\begin{bmatrix} \chi_x^{(j)} \\ \chi_y^{(j)} \end{bmatrix} = \begin{bmatrix} \cos(2\pi j/p) \\ \sin(2\pi j/p) \end{bmatrix} \tag{12}$$

for $j = 0, 1, \cdots, p-1$. Then the following linear complementarity holds

$$(\lambda e + \chi^T v) \ge 0 \ \perp \ \Gamma_t \ge 0; \tag{13a}$$

$$(-\mu\Gamma_z - e^T\Gamma_t) \ge 0 \ \perp \ \lambda \ge 0 \tag{13b}$$

for each phase, i.e. $v = v(t)$ or v^+, $\Gamma_t = \Gamma_t^-$ or Γ_t^+, $\Gamma_z = \Gamma_z^-$ or Γ_z^+, and $\lambda = \lambda^-$ or λ^+, respectively. This enables the linearized impulse cone to reproduce the original one approximately.
During compression the impulse-momentum equation with the approximated impulse is written as

$$H^{-1}(v(t) - v^-) = \left(\hat{z}\Gamma_z^- + \chi\Gamma_t^-\right) \tag{14}$$

where $v^- = v(t^-)$ is the pre-impact velocity. Together with (10) and (13), this leads to

$$\left[\begin{array}{c|ccc} H^{-1} & \hat{z} & -\chi & 0 \\ \hline -\hat{z}^T & 0 & 0 & 0 \\ \chi^T & 0 & 0 & e \\ 0 & \mu & -e^T & 0 \end{array}\right] \left[\begin{array}{c} v(t) \\ \hline -\Gamma_z^- \\ \Gamma_t^- \\ \lambda^- \end{array}\right] - \left[\begin{array}{c} H^{-1}v^- \\ \hline 0 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -v_z(t) \\ \sigma^- \\ \zeta^- \end{array}\right];$$

$$\begin{bmatrix} -v_z(t) \\ \sigma^- \\ \zeta^- \end{bmatrix} \ge 0 \quad \perp \quad \begin{bmatrix} -\Gamma_z^- \\ \Gamma_t^- \\ \lambda^- \end{bmatrix} \ge 0.$$

The impulse-momentum equation during expansion phase is expressed as

$$H^{-1}(v^{+} - v(t)) = \left(\widehat{z}\widetilde{\Gamma}_z + \chi\Gamma_t^{+}\right) + \epsilon\widehat{z}\Gamma_z^{-}. \tag{15}$$

The impulse due to restitution, $\epsilon\Gamma_z^{-}$ acts as an external impulse. Coupled with the complementarity conditions, the system is described by

$$\left[\begin{array}{c|ccc} H^{-1} & \widehat{z} & -\chi & 0 \\ \hline -\widehat{z}^T & 0 & 0 & 0 \\ \chi^T & 0 & 0 & e \\ 0 & \mu & -e^T & 0 \end{array}\right]\left[\begin{array}{c} v^{+} \\ \hline -\widetilde{\Gamma}_z \\ \Gamma_t^{+} \\ \lambda^{+} \end{array}\right] - \left[\begin{array}{c} H^{-1}v(t) + \epsilon\widehat{z}\Gamma_z^{-} \\ \hline 0 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline -v_z^{+} \\ \sigma^{+} \\ \zeta^{+} \end{array}\right];$$

$$\begin{bmatrix} -v_z^{+} \\ \sigma^{+} \\ \zeta^{+} \end{bmatrix} \geq 0 \quad \perp \quad \begin{bmatrix} -\widetilde{\Gamma}_z \\ \Gamma_t^{+} \\ \lambda^{+} \end{bmatrix} \geq 0,$$

Eliminating noncomplementary variable $v(t)$ and $v(t^{+})$ from each mixed LCP given above yields the following LCPs

$$\begin{bmatrix} -v_z(t) \\ \sigma^{-} \\ \zeta^{-} \end{bmatrix} = M^{-}\begin{bmatrix} -\Gamma_z^{-} \\ \Gamma_t^{-} \\ \lambda^{-} \end{bmatrix} + b^{-}; \tag{16a}$$

$$\begin{bmatrix} -v_z^{+} \\ \sigma^{+} \\ \zeta^{+} \end{bmatrix} = M^{+}\begin{bmatrix} -\Gamma_z^{+} \\ \Gamma_t^{+} \\ \lambda^{+} \end{bmatrix} + b^{+}; \tag{16b}$$

where

$$M^{-} = M^{+} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ \mu & -e^T & 0 \end{bmatrix} + \begin{bmatrix} -\widehat{z}^T \\ \chi^T \\ 0 \end{bmatrix} H\begin{bmatrix} -\widehat{z} & \chi & 0 \end{bmatrix};$$

$$b^{-} = \begin{bmatrix} -\widehat{z}^T \\ \chi^T \\ 0 \end{bmatrix} v^{-}; \qquad b^{+} = \begin{bmatrix} -\widehat{z}^T \\ \chi^T \\ 0 \end{bmatrix}\left(v(t) + \epsilon H\widehat{z}\Gamma_z^{-}\right).$$

It was shown that these two LCPs always have a solution which can be computed efficiently by Lemke's complementary pivoting algorithm. [9], [8], [7]

3. Mode Evolution by Algebraic Equations

Suppose that collision begins at time t^{-} and finishes at t^{+}. For any intermediate period $[\tau_1, \tau_2]$ such that $t^{-} \leq \tau_1 < \tau_2 \leq t^{+}$, the behavior is partially governed by the impulse-momentum equation $IM(\tau_1 : \tau_2)$

The method of mode evolution by LCP progresses the collision mode across only two modes, i.e. compression and expansion. As such, the method do not care if sticking or resumed sliding occur during each mode. To the contrary, the collision mode evolves according to the sliding differential equation in the method of mode evolution by differential equation. However, numerical integration has some undesirable effects, such as much heavier computation.

In this section, we propose a new method to mode evolution, which solves a set of algebraic equations sequentially defining mode changes, called events. Further, the quadratic impulse cone is attained without any approximation. Except the algebraic nature of the method, it is of much similar character as the one by differential equation in the sense that collision mode evolves step by step depending on whether some events happened or not. Each mode is triggered by an event. It depends on events whether a mode is terminated and new mode initiates, or the mode continues. Events are described by their own defining algebraic equations.

3.1. Nonlinear complementarity form of Frictional impulse law

The method adopts the quadratic impulse cone (8). In the LCP based method, an approximated cone constraint is applied only for two modes. To the contrary, the exact cone constraint is applied as many times as the number of evolved modes. It is not a priori determined how many modes would generate without sequentially enumerating every possible event.

Governing equations for each mode should be composed of six equations, three of them already given by the impulse-momentum equations. Additional constitutive equations are provided by mode definition. For example, persistent-sticking mode would entail two equations that v_x and v_y vanish identically during the mode. If sliding prevails during a mode, the mode definition readily produces one condition $\Gamma_x^2 + \Gamma_y^2 = \mu_d^2\Gamma_z^2$.

As expected, mode definitions involve the impulse cone constraints in one way or the other. For unified generation of mode definitions, we propose the following (nonlinear) complementarity formulation between the tangential impulse vector and the sliding velocity vector. We consider a mode which is triggered at τ_1 and extends to τ_2.

$$\begin{aligned} & v_x^2(\tau_2) + v_y^2(\tau_2) \geq 0; \\ & \quad \perp \quad \mu_d^2\Gamma_z^2(\tau_1 : \tau_2) - \Gamma_x^2(\tau_1 : \tau_2) - \Gamma_y^2(\tau_1 : \tau_2) \geq 0. \end{aligned} \tag{17}$$

Being in this form, it is not a standard nonlinear complementarity form [9].

It should be observed that two enumerative cases resulting from the complementarity are not symmetric in number of equations it generates. The first case $v_x^2(\tau_2) + v_y^2(\tau_2) = 0$ yields two equations $v_x(\tau_2) = v_y(\tau_2) = 0$, whereas the second gives only one equation $\mu^2\Gamma_z^2(\tau_1 : \tau_2) = \Gamma_x^2(\tau_1 : \tau_2) + \Gamma_y^2(\tau_1 : \tau_2)$. This unbalance is dealt with by the following algebraic equation and inequality

$$\Gamma_x(\tau_1 : \tau_2)v_y(\tau_2) - \Gamma_y(\tau_1 : \tau_2)v_x(\tau_2) = 0 \tag{18a}$$

$$\Gamma_x(\tau_1 : \tau_2)v_x(\tau_2) + \Gamma_y(\tau_1 : \tau_2)v_y(\tau_2) \leq 0. \tag{18b}$$

They are concerned with the direction of impulse vector when the mode finishes at τ_2.[1]

3.2. Mode evolution by event enumeration

It has been mentioned that it is determined how a mode evolves by enumerating every possible event. Possible mode evolutions are illustrated by Fig. 1.

Suppose that $v_x^2(t^{-}) + v_y^2(t^{-}) > 0$, sliding at t^{-}. Then, the mode **CSlide** (Compression-Sliding) is initiated. Now

[1] In particular, if $v_x^2(\tau_2) + v_y^2(\tau_2) \neq 0$, they are equivalent to the single condition $\mathrm{atan2}(\Gamma_y, \Gamma_x) - \mathrm{atan2}(v_y, v_x) = \pi$, which stipulates that the tangential impulse vector is opposite to the sliding velocity vector. However, when $v_x^2(\tau_2) + v_y^2(\tau_2) = 0$, this does not hold, as $\mathrm{atan2}(\Gamma_y, \Gamma_x) = \pi$. To the contrary, the proposed conditions (18a) and (18b) are trivially satisfied.

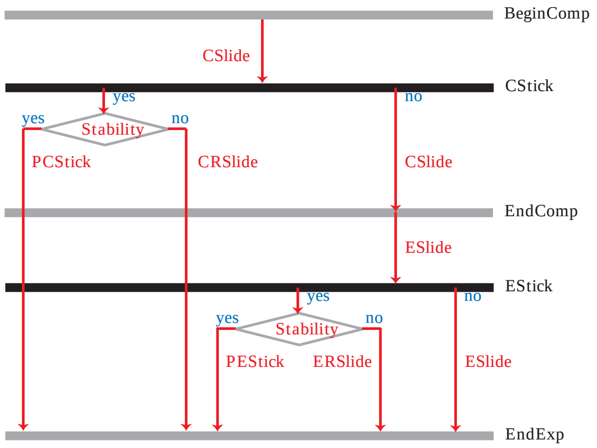


Fig. 1. Mode evolution by event enumeration

the events **CStick** (Compression-Sticking) is checked using its governing equations. If the occurrence is denied, then the event **EndComp** (End-of-Compression) is triggered and the mode **CSlide** finishes and the new mode **ESlide** (Expansion-Sliding) begins. Now, the event **EStick** (Expansion-Sticking) is examined of its existence. If it does not exist, then the mode continues to the event **EndExp** (End-of-Expansion). Were the event **EStick** affirmed, the stability of sticking should be examined. A New mode **PEStick** (Persistent-**EStick**) begins for stable sticking, while the mode **ERSlide** (Expansion-Resumed-Sliding) initiates for unstable sticking. Then either method prevails until the event **EndExp**. In case of the event **CStick**, either the mode **PCstick** (Persistent-**CStick**) or the mode **CRSlide** (Compression-Resumed-Sliding) follows depending on the stability of **CStick**. Then, either mode continues through the events **EndComp** and **EndExp**.

3.2.1 Governing equations for the event **CStick**

While in the mode **CSlide**, the event **CStick** is affirmed in the case of the existence of a time instant $\tau \leq t$ such that $v_x^2(\tau) + v_y^2(\tau) = 0$, while $v_x^2(\sigma) + v_y^2(\sigma) > 0$ for $\forall \sigma \in [t^-, \tau)$. The condition is readily expressed by the following two equations in light of (18)

$$v_x(\tau) = v_y(\tau) = 0; \tag{19a}$$

and

$$\mu_d^2 \Gamma_z^2(t^- : \tau) = \Gamma_x^2(t^- : \tau) + \Gamma_y^2(t^- : \tau). \tag{19b}$$

The latter expresses the property that sliding has prevailed just before **CStick**. These three equations complement the impulse-momentum equation $IM(t^- : \tau)$. It should be noted that the existence of such hypothetical τ is not yet validated. After substituting (19a) to $IM_x(t^- : \tau)$ and $IM_y(t^- : \tau)$ and solving for $\Gamma_x(t^- : \tau)$ and $\Gamma_y(t^- : \tau)$, the intermediate result being plugged into $IM_z(t^- : \tau)$ yields the normal velocity at τ, the **CStick** instant, by

$$v_z(\tau) = \frac{D_{31}v_x(t^-) + D_{32}v_y(t^-) + D_{33}v_z(t^-)}{D_{33}} + \frac{D}{D_{33}}\Gamma_z(t^- : \tau), \tag{20}$$

where

$$D_{31} = UV - YW; D_{32} = UW - XV; D_{33} = XY - U^2.$$

Later, the following subdeterminants are also of interest

$$D_{11} = YZ - V^2; D_{21} = WV - UZ; D_{22} = XZ - W^2$$

with the following determinant

$$D = WD_{31} + VD_{32} + ZD_{33}.$$

By substituting the intermediate $\Gamma_x(t^- : \tau)$ and $\Gamma_y(t^- : \tau)$ in (19b), the governing equations for the existence of **CStick** reduce to the quadratic equation in $\Gamma_z(t^- : \tau)$

$$q(\Gamma_z(t^- : \tau)) := Q\Gamma_z(t^- : \tau)^2 + L\Gamma_z(t^- : \tau) + C = 0 \tag{21}$$

with

$$\begin{aligned} Q &= D_{33}^2 \left(\mu_d^2 - \lambda\right) \\ L &= 2((D_{31}Y - D_{32}U)v_x(t^-) - (D_{31}U - D_{32}X)v_y(t^-)) \\ C &= -(Yv_x(t^-) - Uv_y(t^-))^2 - (Uv_x(t^-) - Xv_y(t^-))^2 \end{aligned}$$

where

$$\lambda = \frac{D_{31}^2 + D_{32}^2}{D_{33}^2}. \tag{22}$$

The inertia related quantity λ is used to check the stability of sticking, as shown below.
Since **CStick** should occur no later than the event **EndComp**, the candidate solution should satisfy

$$-\Gamma_z(t^- : \tau) \geq 0; \qquad -v_z(\tau) \leq 0.$$

These inequalities are called the *event hypotheses*. They should be passed to guarantee the existence of **CStick**. In view of (20), the assumption hypotheses are succinctly expressed as

$$0 \leq -\Gamma_z(t^- : \tau) \leq \gamma, \tag{23}$$

where

$$\gamma = \frac{D_{31}v_x(t^-) + D_{32}v_y(t^-) + D_{33}v_z(t^-)}{D}. \tag{24}$$

Among the solutions to (21), only the one satisfying (23) is feasible.

3.2.2 Sticking Stability

Suppose that the event **CStick** has occurred. This signals the termination of the mode **CSlide**. One has to examine whether sliding would resume or sticking would continue to hold. This can be confirmed by examining whether the line of sticking lies inside the friction cone [4]. The line of sticking is defined by

$$\begin{aligned} -v_x(t^-) &= X\Gamma_x(t^- : \sigma) + U\Gamma_y(t^- : \sigma) + W\Gamma_z(t^- : \sigma) \\ -v_y(t^-) &= U\Gamma_x(t^- : \sigma) + Y\Gamma_y(t^- : \sigma) + V\Gamma_z(t^- : \sigma) \end{aligned}$$

as $v_x(\sigma) = v_y(\sigma) = 0$. It is readily verified that the angle θ it makes with the Γ_z-axis is such that

$$\tan\theta = \frac{\sqrt{(YW - UV)^2 + (U^TW - XV)^2}}{-XY + UU^T}.$$

The friction cone has the angle α such that $\tan\alpha = \mu_s$ for the static friction coefficient μ_s. One can see that if

$$\frac{(YW-UV)^2+(UW-XV)^2}{(XY-U^2)^2} = \lambda \leq \mu_s^2 \tag{25}$$

where λ is the one defined by (22), the line of sticking lies within the friction cone from the event **CStick**. Otherwise, sliding should resume. It is worth noting that it depends only on the inertial parameter, λ, and the static friction coefficient whether sticking is stable or not. The condition (25) is identical to the one given by Mirtich [2, Ch. 3, Thm. 9].

3.2.3 The mode **PCStick**

If $\lambda \leq \mu_s^2$, then sticking remains persistent, till the end of expansion phase. This mode is called **PCStick**.

During this mode, the event **EndComp** is defined by

$$v_z(t)=0; \quad v_x(t)=v_y(t)=0. \tag{26}$$

These three conditions together with $IM(\tau:t)$ determine completely the state at **EndComp**. When eliminating $\Gamma_x(\tau:t)$ and $\Gamma_y(\tau:t)$ from $IM_x(\tau:t)$ and $IM_y(\tau:t)$, then $IM_z(\tau:t)$ yields

$$\Gamma_z(\tau:t) = -\frac{D_{33}}{D}v_z(\tau).$$

Substituting $v_z(\tau)$ from (20), we get

$$\Gamma_z(\tau:t) = -\Gamma_z(t^-:\tau) - \frac{D_{31}v_x(t^-)+D_{32}v_y(t^-)+D_{33}v_z(t^-)}{D}. \tag{27}$$

The normal impulse for the whole compression phase, i.e. $\Gamma_z(t^-:t) = \Gamma_z(t^-:\tau)+\Gamma_z(\tau:t)$, in **PCStick** is given by

$$\Gamma_z(t^-:t) = -\frac{D_{31}v_x(t^-)+D_{32}v_y(t^-)+D_{33}v_z(t^-)}{D}. \tag{28}$$

In view of (24), one can see that $\Gamma_z(t^-:t) = -\gamma$.

The event **EndExp** in **PCStick** is defined by $v_x(t^+) = v_y(t^+) = 0$ (because of **PCStick**) and the termination condition due to Poisson's law of impulse $\Gamma_z(t:t^+) = \epsilon\Gamma_z(t^-:t)$. Solving $IM(t:t^+)$ for the post-impact normal velocity $v_z(t^+)$, we get

$$v_z(t^+) = \frac{D}{D_{33}}\Gamma_z(t:t^+) = \epsilon\frac{D}{D_{33}}\Gamma_z(t^-:t). \tag{29}$$

Making use of (28) leads to

$$v_z(t^+) = -\frac{\epsilon(D_{31}v_x(t^-)+D_{32}v_y(t^-)+D_{33}v_z(t^-))}{D_{33}}. \tag{30}$$

3.2.4 The mode **CRSlide**

Despite the event **CStick**, sliding resumes if the friction coefficient is not large enough in the sense of (25).

The mode **CRSlide** first encounters the event **EndComp** defined by $v_z(t)=0$ with $IM(\tau:t)$

$$v_x(t) = X\Gamma_x(\tau:t)+U\Gamma_y(\tau:t)+W\Gamma_z(\tau:t) \tag{31a}$$
$$v_y(t) = U\Gamma_x(\tau:t)+Y\Gamma_y(\tau:t)+V\Gamma_z(\tau:t) \tag{31b}$$
$$-v_z(\tau) = W\Gamma_x(\tau:t)+V\Gamma_y(\tau:t)+Z\Gamma_z(\tau:t) \tag{31c}$$

Because sliding prevails at the end of compression, the impulse cone complementarity (17) and (18a) are resolved by

$$0 = \mu_d^2\Gamma_z^2(\tau:t)-\Gamma_x^2(\tau:t)-\Gamma_y^2(\tau:t) \tag{31d}$$
$$0 = \Gamma_x(\tau:t)v_y(t)-\Gamma_y(\tau:t)v_x(t). \tag{31e}$$

Eliminating $\Gamma_z(\tau:t)$ form (31c)

$$\Gamma_z(\tau:t) = -\frac{W}{Z}\Gamma_x(\tau:t)-\frac{V}{Z}\Gamma_y(\tau:t)-\frac{1}{Z}v_z(\tau), \tag{32}$$

reduces the impulse cone magnitude condition given by (31d) to a conic equation in the tangential impulse domain $(\Gamma_x(\tau:t),\Gamma_y(\tau,t))$

$$\begin{aligned}0 &= (\mu_d^2W^2-Z^2)\Gamma_x^2(\tau:t)+(\mu_d^2V^2-Z^2)\Gamma_y^2(\tau:t)\\&+2\mu_d^2VW\Gamma_x(\tau:t)\Gamma_y(\tau:t)+2\mu_d^2Wv_z(\tau)\Gamma_x(\tau:t)\\&+2\mu_d^2Vv_z(\tau)\Gamma_y(\tau:t)+\mu_d^2v_z^2(\tau).\end{aligned} \tag{33}$$

Further eliminating $v_x(t)$ and $v_y(t)$ from (31a) and (31b)

$$v_x(t) = \frac{D_{22}}{Z}\Gamma_x(\tau:t)-\frac{D_{21}}{Z}\Gamma_y(\tau:t)-\frac{W}{Z}v_z(\tau) \tag{34a}$$
$$v_y(t) = -\frac{D_{21}}{Z}\Gamma_x(\tau:t)+\frac{D_{11}}{Z}\Gamma_y(\tau:t)-\frac{V}{Z}v_z(\tau), \tag{34b}$$

the impulse cone direction constraint, i.e. (31e), yields another conic equation

$$\begin{aligned}0 &= -D_{21}\Gamma_x^2(\tau,t)+D_{21}\Gamma_y^2(\tau,t)+(D_{11}-D_{22})\Gamma_x(\tau:t)\Gamma_y(\tau:t)\\&-Vv_z(\tau)\Gamma_x(\tau:t)+Wv_z(\tau)\Gamma_y(\tau:t).\end{aligned} \tag{35}$$

The two equations define quadratic curves in the impulse domain $(\Gamma_x(\tau:t),\Gamma_y(\tau,t))$ of the form $A\Gamma_x^2+B\Gamma_y^2+C\Gamma_x\Gamma_y+D\Gamma_x+E\Gamma_y+F=0$. They are solved numerically to yield $(\Gamma_x(\tau:t),\Gamma_y(\tau,t))$. Among possibly more than one solutions, the one satisfying (18b) is the desired one. Then, (28) is used to compute $\Gamma_z(\tau:t)$. The total normal impulse during compression phase is given by $\Gamma_z(t^-:t) = \Gamma_z(t^-:\tau)+\Gamma_z(\tau:t)$, where $\Gamma_z(t^-:\tau)$ is computed previously in determining **CStick**.

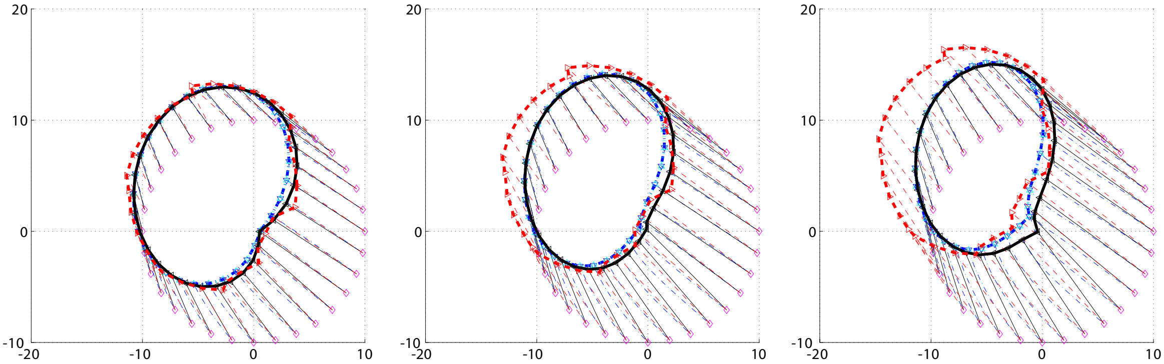
Next the **EndExp** is determined by solving $IM(t:t^+)$ with $\Gamma_z(t:t^+)$ given by $\Gamma_z(t:t^+) = \epsilon\Gamma_z(t^-:t)$. As sliding prevails at the end of expansion, the sliding velocity vector at t^+ and the tangential impulse vector satisfies

$$0 = \mu_d^2\Gamma_z^2(t:t^+)-\Gamma_x^2(t:t^+)-\Gamma_y^2(t:t^+)$$
$$0 = \Gamma_x(t:t^+)v_y(t^+)-\Gamma_y(t:t^+)v_x(t^+),$$

In the impulse domain $(\Gamma_x(t:t^+),\Gamma_y(t:t^+))$ they define the following conics

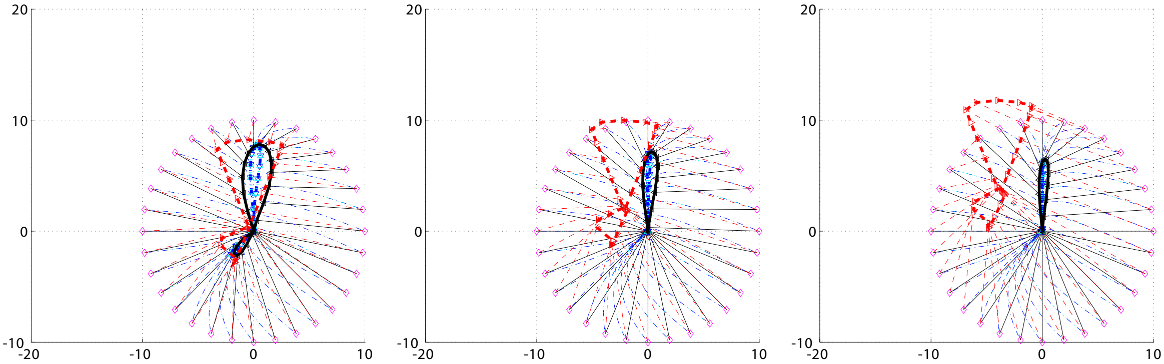
$$0 = \Gamma_x^2(t:t^+)+\Gamma_y^2(t:t^+)-\mu_d^2\epsilon^2\Gamma_z^2(t^-:t) \tag{36}$$
$$\begin{aligned}0 &= U\Gamma_x^2(t:t^+)-U\Gamma_y^2(t:t^+)+(Y-X)\Gamma_x(t:t^+)\Gamma_y(t:t^+)\\&+(\epsilon V\Gamma_z(t^-:t)+v_y(t))\Gamma_x(t:t^+)\\&-(\epsilon W\Gamma_z(t^-:t)+v_x(t))\Gamma_y(t:t^+).\end{aligned} \tag{37}$$

Solving these equations yields the tangential impulse at t^+, which being substituted to $IM(t:t^+)$ generates the post-impact sliding velocity and normal velocity.



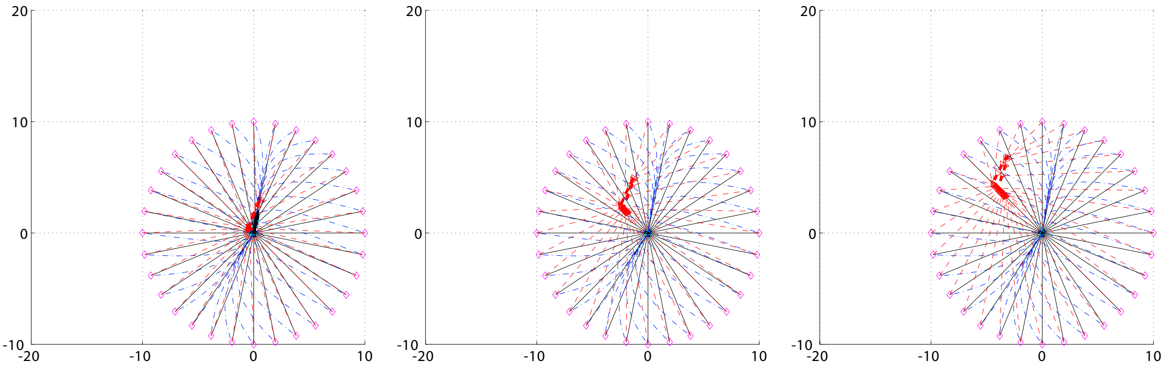
(a) $\epsilon = 0.1$ (b) $\epsilon = 0.5$ (c) $\epsilon = 0.9$

Fig. 2. Collision resolution for $\mu = 0.1$ ('–' : Park's/'- -': Anitescu's/'-·-': Routh's)



(a) $\epsilon = 0.1$ (b) $\epsilon = 0.5$ (c) $\epsilon = 0.9$

Fig. 3. Collision resolution for $\mu = 0.5$ ('–' : Park's/'- -': Anitescu's/'-·-': Routh's)



(a) $\epsilon = 0.1$ (b) $\epsilon = 0.5$ (c) $\epsilon = 0.9$

Fig. 4. Collision resolution for $\mu = 1.0$ ('–' : Park's/'- -': Anitescu's/'-·-': Routh's)

3.2.5 Remaining modes and events

The remaining modes can be determined by event enumeration using the same technique as described above.

4. Numerical Simulation

Consider a single collision with $H = \begin{bmatrix} 8 & -2 & 1 \\ -2 & 3 & -1 \\ 1 & -1 & 5 \end{bmatrix} (kg^{-1})$ with $v_z(t^-) = 15(m/sec)$. Initial sliding velocity $v_x(t^-)$ and $v_y(t^-)$ are sampled by

$$\begin{bmatrix} v_x(t^-) \\ v_y(t^-) \end{bmatrix} = \begin{bmatrix} \beta\cos(2\pi(j-1)/32) \\ \beta\sin(2\pi(j-1)/32) \end{bmatrix} (m/sec)$$

with $\beta = 10(m/sec)$ for $k = 1, 2, \cdots, 32$. To examine a greater spectrum of collision resolution behavior, a variety of combinations of collision parameters, consisting of $\mu = \mu_d = \mu_s$, the friction coefficient, and ϵ, coefficient of restitution, were employed from $\mu = 0.1$ (low friction), 0.5

(moderate friction), 1.0 (high friction) and from $\epsilon = 0.1$ (somewhat inelastic), 0.5 (partially elastic), 0.9 (somewhat elastic). All three collision resolution methods were simulated: Routh's method, Anitescu's method, and the proposed method referred to as Park's method.

Figs. 2, 3, 4 summarized the simulation result for almost frictionless case ($\mu = 0.01$), light friction case ($\mu = 0.1$), moderate friction case ($\mu = 0.5$), and high friction case ($\mu = 1.0$), respectively. For each figure, subfigures (a), (b), and (c) corresponds to inelastic case ($\epsilon = 0.1$), partially elastic case ($\epsilon = 0.5$), and elastic case ($\epsilon = 0.9$). In the figures, initial sliding velocities are plotted as diamonds in the figure. The light lines display flows of the sliding velocities for Routh's method, while for Park's and Anitescu's method, both of which are algebraic method in nature, they simply interconnect the initial sliding velocity and the corresponding final sliding velocity. Post-impact velocities are connected by thicker lines for each method. Different line styles are used to indicate the associated method: '–' (black) for Park's, '--' (red) for Anitescu's, '-·-' (blue) for Routh's method. The color in parentheses indicates the color if you see them in color.

As a matter of fact, for frictionless cases all the methods generate virtually same post-impact sliding velocities for all restitution coefficients. The coincidence cannot be observed any more for frictional cases. In particular, Anitescu's method based on LCP deviates conspicuously qualitatively and quantitatively as the restitution becomes more elastic, or coefficient of restitution gets larger. Qualitative difference becomes significant when one compares the behavior shown in Fig. 4 (a), (b), and (c). All the other methods predicted persistent sticking for all initial sliding velocities, while Anitescu's predicted sliding post-impact velocity for every initial velocity. To the contrary, it can be said that Park's method and Routh's method generated quite similar behavior for highly frictional collisions. For moderate friction, shown in Fig. 3, the loci of post-impact velocities for them are quite similar except Park's method produces a slightly more spread results. Numerical evaluation of energy loss would reveal that Park's method generally dissipates a less energy, hence larger sliding velocity, than Routh's method. One can see a more significant difference for light friction case, as shown in Fig. 2.

5. Concluding Remarks

In this article, collision resolution methods were analyzed. It is worth noting that rigid collision models are only approximations of complex physical processes involving wave propagation, nonnegligible flexibility and the like. In some cases, experiments report that the approximation is quite faithful in predicting the behavior summarized with the post-impact velocities [10], which supports the utility of such rigid collision models. However, the approximation can never be universally correct. It becomes clear through numerical experiments that the LCP-based model is not suitable as it cannot handle sticking occurring during collision. The model based on the sliding differential equation seems compatible with the physical process of collision under good identification of collision parameters, mainly consisting of frictional and restitution coefficient. However, it involves numerical integration of differential equations. Our experience tells that as friction coefficient gets larger, it takes longer to tend to sticking, as the equation becomes closer to singularity. One can apply the proposed method as an alternative. It can handle sticking and resumed sliding only by solving a set of algebraic equations, which is less time-consuming. Theoretical validity can be said to be of a same degree as the differential equation based method, if the impulse cone constraint approximates the real impulse vector during collision the same degree as the friction cone constraint approximates the real impulse. Numerical experiments corroborated the validity of the proposed method as an efficient alternative to the differential equation based methods. An experiment is to be conducted which would advocate the predicting capability of the proposed method of real collision phenomena.

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