Delay-dependent stabilization for time-delay systems: An LMI approach*

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Abstract: This paper focuses on the problem of asymptotic stabilization for time-delay systems. To this end, a memoryless state feedback controller is proposed. Then, based on the Lyapunov method, a delay-dependent stabilization criterion is devised by taking the relationship between the terms in the Leibniz-Newton formula into account. Certain free weighting matrices are used to express this relationship and linear matrix inequalities (LMIs)-based algorithm to design the controller stabilizing the system.

Keywords: Time-delay systems, Delay-dependent criterion, LMI, Lyapunov method.

1. Introduction

Time-delay leads to instability and poor performance in various engineering systems such as chemical process, electrical network, nuclear reactor, biological system and economic systems. Thus, the problems of asymptotic stability and stabilization for time-delay systems have received considerable attention over the decades.

In the past decades, Lyapunov method, characteristic equation method, or the state solution approach have been utilized for analyzing the problems (for details, see [2]). In recent years, LMI approaches based on convex optimization algorithms have been extensively applied to solve the problems. The LMI approach needs no turning of parameters and / or matrices to derive certain criteria for stability and stabilization of various dynamic system. Also it can be solved numerically efficiently by using interior-point algorithm which has recently been developed for solving optimization problem involving LMIs. Thus, using the LMI approach, many stability criteria for guaranteeing stability of the system developed in the literature. In general, the criteria can be classified in two categories: delay-independent and delay-dependent criteria. One approach is to contrive the stability conditions which do not dependent on the delay, and the other is to take it into account. Since the delay-dependent criteria make use of information on the length of delays, they are usually less conservative than delay-independent systems (see [3], [4], and [5]). Recently, Wu *et al.* [5] introduced a new lemma to derive stability criterion for the system, and they showed their results which are less conservative then another one. However, the design problem of stabilizing controller is not considered in their work.

This paper concerned with the problem of delay-dependent asymptotic stabilization of time-delay systems utilizing the method developed by [5]. We propose a memoryless state feedback controller which maximizes the delay bound of the system. A novel stabilization criterion is derived using Lyapunov theory and LMI framework. In this paper, in order to solve the LMIs, we utilize Matlab's LMI Control Toolbox, which implements state-of the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithms [1]. A numerical example is given to illustrate our main method.

Notations. $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices,and $\star$ represents the elements below the main diagonal of a symmetric block matrix. $diag\{\cdot\}$ denotes the block diagonal matrix. The notation $W > 0$ $(\geq, <, \leq 0)$ denotes a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix W.

2. Problem formulation

Consider the following delayed system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-d) + Bu(t), \quad t > 0, \\ x(t) &= \phi(t), \quad t \in [-d, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $u(k) \in \mathbb{R}^m$ is the control input, $A, A_1 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, $\phi(t)$ is the initial condition of $t \in [-d, 0]$, and $d > 0$ denotes the constant time delay.

In this paper, to stabilize system (1), we propose the following memoryless state-feedback controller

$$u(t) = Kx(t) \tag{2}$$

where $K \in \mathbb{R}^{m\times n}$ is a constant gain matrix to be designed later.

The following lemma will be used to proof our main theorem.

Lemma 1. [5] The free weighting matrices Y and T indicate the relationship between the terms in the Leibniz-Newton formula.

$$\begin{aligned} &2(x^T(t)Y + x^T(t-d)T) \\ &\quad \times \Big(x(t) - \int_{t-d}^{t} \dot{x}(s)ds - x(t-d) \Big) = 0. \end{aligned} \tag{3}$$

3. Main results

In this section, we present a delay-dependent criterion for asymptotic stabilization of time-delay system (1) based on the Lyapunov method and LMI approach.

Theorem 1. Given $d > 0$, the system (1) with the control input $u(t) = GV^{-1}x(t)$ is asymptotically stable if there exist positive-definite matrices $V = V^T > 0, N = N^T > 0$, symmetric matrices G, L, M, and semi-positive-definite matrix $\Sigma =$

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$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \geq 0$ such that

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & dVA^T + dG^TB^T \\ \star & \Pi_{22} & dVA_1^T \\ \star & \star & -dV \end{bmatrix} < 0, \quad (4)$$

$$\Gamma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & L \\ \star & \Sigma_{22} & M \\ \star & \star & V \end{bmatrix} \geq 0, \quad (5)$$

where

$$\begin{aligned} \Pi_{11} &= VA^T + G^TB^T + AV + BG + L + L^T + N + d\Sigma_{11} \\ \Pi_{12} &= A_1V - L + M^T + d\Sigma_{12} \\ \Pi_{22} &= -M - M^T - N + d\Sigma_{22}. \end{aligned}$$

Proof. The closed-loop system of (1) with the control input (2) is

$$\dot{x}(t) = (A + BK)x(t) + A_1x(t-d). \quad (6)$$

Consider a Lyapunov function candidate as

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + \int_{t-d}^{t} x^T(s)Qx(s)ds \\ &\quad + \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)P\dot{x}(s)dsd\theta. \end{aligned} \quad (7)$$

By the Leibniz-Newton formula, the following equation satisfies

$$x(t) - \int_{t-d}^{t} \dot{x}(s)ds - x(t-d) = 0.$$

Then, for any appropriately dimensioned matrices Y and T, Eq. (3) holds. For any semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, the following also holds

$$d\xi^T(t)X\xi(t) - \int_{t-d}^{t} \xi^T(t)X\xi(t)ds = 0, \quad (8)$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-d) \end{bmatrix}^T$.
Then, the time derivative, $\dot{V}(x_t)$, of $V(x_t)$ in (7) is

$$\begin{aligned} \dot{V}(x_t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Qx(t) \\ &\quad - x^T(t-d)Qx(t-d) + d\dot{x}^T(t)P\dot{x}(t) \\ &\quad - \int_{t-d}^{t} \dot{x}^T(s)P\dot{x}(s)ds \\ &= x^T(t)((A+BK)^TP + P(A+BK))x(t) \\ &\quad + 2x^T(t)PA_1x(t-d) + x^T(t)Qx(t) \\ &\quad - x^T(t-d)Qx(t-d) + x^T(t)(d(A+BK)^TP \\ &\quad \times (A+BK))x(t) + x^T(t)(d(A+BK)^TPA_1) \\ &\quad \times x(t-d) + x^T(t-d)(dA_1^TP(A+BK))x(t) \\ &\quad + x^T(t-d)dA_1^TPA_1x(t-d) \\ &\quad - \int_{t-d}^{t} \dot{x}^T(s)P\dot{x}(s)ds. \end{aligned}$$

Utilizing (3) and (8) gives that

$$\begin{aligned} \dot{V}(x_t) &= x^T(t)((A+BK)^TP + P(A+BK))x(t) \\ &\quad + 2x^T(t)PA_1x(t-d) + x^T(t)Qx(t) \\ &\quad - x^T(t-d)Qx(t-d) + x^T(t)(d(A+BK)^TP \\ &\quad \times (A+BK))x(t) + x^T(t)(d(A+BK)^TPA_1) \\ &\quad \times x(t-d) + x^T(t-d)(dA_1^TP(A+BK))x(t) \\ &\quad + x^T(t-d)dA_1^TPA_1x(t-d) \\ &\quad - \int_{t-d}^{t} \dot{x}^T(s)P\dot{x}(s)ds \\ &\quad + 2(x^T(t)Y + x^T(t-d)T) \\ &\quad \times (x(t) - \int_{t-d}^{t} \dot{x}(s)ds - x(t-d)) \\ &\quad + d\xi^T(t)X\xi(t) - \int_{t-d}^{t} \xi^T(t)X\xi(t)ds \\ &= \xi^T(t)\Pi_1\xi(t) - \int_{t-d}^{t} \zeta^T(t,s)\Gamma_1\zeta(t,s)ds, \end{aligned} \quad (9)$$

where

$$\zeta(t,s) = \begin{bmatrix} x^T(t) & x^T(t-d) & \dot{x}^T(s) \end{bmatrix}^T,$$

$$\Pi_1 = \begin{bmatrix} (1,1) & (1,2) \\ \star & (2,2) \end{bmatrix}, \quad (10)$$

$$\Gamma_1 = \begin{bmatrix} X_{11} & X_{12} & Y \\ \star & X_{22} & T \\ \star & \star & P \end{bmatrix}, \quad (11)$$

with

$$\begin{aligned} (1,1) &= (A+BK)^TP + P(A+BK) + Y + Y^T + Q \\ &\quad + dX_{11} + d(A+BK)^TP(A+BK), \\ (1,2) &= PA_1 - Y + T^T + dX_{12} + d(A+BK)^TPA_1, \\ (2,2) &= -T - T^T - Q + dX_{22} + dA_1^TPA_1. \end{aligned}$$

If $\Pi_1 < 0$, and $\Gamma_1 \geq 0$, then $\dot{V}(x_t) < 0$ for any $\xi(t) \neq 0$, which guarantees the stability of the system (1)
By Schur complement [1], $\Pi_1 < 0$ is equivalent to

$$\Pi_2 = \begin{bmatrix} \begin{matrix}(A+BK)^TP \\ +P(A+BK) \\ +Y+Y^T \\ +Q+dX_{11}\end{matrix} & \begin{matrix}PA_1 - Y \\ +T^T + dX_{12}\end{matrix} & d(A+BK)^TP \\ \star & \begin{matrix}-T-T^T \\ -Q+dX_{22}\end{matrix} & dA_1^TP \\ \star & \star & -dP \end{bmatrix} < 0. \quad (12)$$

By pre- and post-multiplying $diag\{P^{-1}, P^{-1}, P^{-1}\}$ for both sides of Π_2 and Γ_1, respectively, and defining $V = P^{-1}$, $L = P^{-1}YP^{-1}$, $M = P^{-1}TP^{-1}$, $N = P^{-1}QP^{-1}$, $\Sigma_{11} = P^{-1}X_{11}P^{-1}$, $\Sigma_{12} = P^{-1}X_{12}P^{-1}$, $\Sigma_{22} = P^{-1}X_{11}P^{-1}$, $G = KP^{-1}$, we have the two inequalities (4) and (5). This gives that the system (1) with $u(t) = GV^{-1}$ is asymptotically stabilized. This completes the proof. ■

Remark 1. The proposed method can be easily extended to time-delay system with time-varying delay $d(t)$ satisfying $\dot{d}(t) \leq 1$.

4. Numerical example

This section gives an example to illustrate our result in this paper. Consider the system studied in [6], [7], [8]

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix} x(t-d) \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).\end{aligned}$$

Now, we address the problem of finding a state-feedback controller for guaranteeing stability of the above system.

Table 1 gives a comparison of several results about the maximum allowable bound of delay and corresponding control gain.

Table 1. Stability bound of d and control gain K

	d	Control gain
Li & Souza [6]	$d \leq 0.999$	$-[0.10452 \quad 749058]$
Fridman & Shaked [7][8]	$d \leq 1.408$	$-[53.51 \quad 294.935]$
Ours	$d \leq 1.6$	$-[0.001 \;\; 1.0154]$

From Table 1, One can see that our result guarantees larger delay bound and small control gain.

5. Conclusion

This paper addressed the problems of asymptotic stabilization for time-delay systems. An LMI-based method for delay-dependent stabilization via linear memoryless state feedback has been developed. The robust stabilization of uncertain time-delay systems will be discussed in our future works.

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