An Accurate Estimation of a Modal System with Initial Conditions (ICCAS 2004)

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Abstract: In this paper, we propose the AWLS/MFT (Adaptive Weighed Least Squares/ Modulation Function Technique) devised by A. E. Pearson *et al.* for the transfer function estimation of a modal system and investigate the performance of several algorithms, the Gram matrix method, a Luenberger Observer (LO), Least Squares (LS), and Recursive Least Squares (RLS), for the estimation of initial conditions. With the benefit of the Modulation Function Technique (MFT), we can separate the estimation problem into two phases: the transfer function parameters are estimated in the first phase, and the initial conditions are estimated in the second phase. The LO method produces excellent IC estimates in the noise free case, but the other three methods show better performance in the noisy case. Finally, we compared our result with the Prony based method. In the noisy case, the AWLS and one of the three methods - Gram matrix, LS, and RLS- show better performance in the output Signal to Error Ratio (SER) aspect than the Prony based method under the same simulation conditions.

Keywords: Prony method, AWLS/MFT, Modal system, Initial condition, Luenberger Observer

1. INTRODUCTION

Some 200 years ago, Prony [1] proposed a basic signal analysis method to approximate a signal by a weighted sum of *n* exponentials. In other words, a real signal $y(t)$ that can be approximated by:

$$\hat{y}(t) = \sum_{i=1}^{n} B_i e^{\lambda_i t} \tag{1}$$

for continuous time t ≥ 0, where $B_i \in C,\ \ i = 1, 2, \cdots, n,$ are the output residues, and $\lambda_i \in C$ are the continuous-time eigenvalues. The Prony approach uses two separate least-squares solutions, each of order *n*, with the first least-squares solution resulting in the eigenvalues of the signal, and the second yielding the weighting terms (or residues) in the summation.

However, direct application of the Prony method is very limited. **First**, the method is known to be exceptionally sensitive to measurement errors in the data samples [5]. **Second**, the algorithm requires *a priori* knowledge of the model order. If the model is not known and the data are contaminated by noise, an overmodelled polynomial may be used. The oversized model results in an estimation of extraneous modes differing with the physical modes. **Third**, in addition to the input signal being restricted to be of a special form, the Prony approach has the disadvantage that typically not all the available input-output data is used in forming the estimates. In signal processing areas, the limitations of the Prony method have been recognized and many remedies have been suggested [6] [7]. In this paper, we suggest the AWLS/MFT in [8] for the estimation of a parameterized transfer function, which is very tolerant to noise. In the AWLS/MFT algorithm, the Modulation Function Technique (MFT) devised by Shinbrot [9] converts the differential equation into an algebraic equation, which makes it easier to solve the identification. Shinbrot's MFT avoids dealing with the unknown initial conditions over each time interval $[t_i, t_{i+1}]$, and avoids differentiating the original data. Moreover, using the regression error covariance, which is a function of unknown parameters, as a weighting matrix, and the method of successive iteration, the AWLS greatly improved estimation performance [10]. With the beneficiary of the MFT, the system identification problem can be separated into two phases: transfer function (eigenvalues and transfer-function residues) is estimated in the first phase, and initial condition residues are estimated in the second phase. In addition to the AWLS, we propose the Gram matrix method (GRAM) [11], a Luenberger Observer (LO) [12], well-known Least Squares (LS), and Recursive Least Squares (RLS) for the estimation of ICs in the second phase. The main objective of this chapter is the investigation of the IC estimation characteristics for the several methods. In Section 2, system model and system matrix transformation are introduced. Section 3 contains a description of the several identification methods applied for ICs estimation in this work. In Section 4, the IC estimation characteristics using LO are investigated with no noise. A comparison of estimation performance with noisy output data is presented in Section 5, prior to the concluding section, Section 6.

2. SYSTEM CHARACTERIZATION

In this section, the system form to be identified is described and the model transformation is introduced.

2.1 System Model

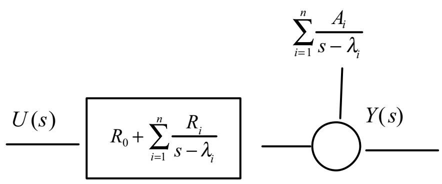


Fig. 1 System model in parallel form

The system model shown in Fig. 1 drawn from [4], is a single-input single-output system with Laplace transform represented in standard parallel form:

$$G(s) = \left(R_0 + \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} \right) \tag{2}$$

In Fig. 1, the initial condition terms are included explicitly in the summation preceding the output $y(t)$, so that the input $u(t)$ can be taken as 0 for $t < 0$. The λ_i's are the eigenvalues of the system, R_0 is a feed through gain, R_1 through R_n are the system residues and A_1 through A_n are the initial condition residues. The λ_i s are assumed to be distinct and can occur in complex conjugate pairs. Residues corresponding to complex conjugate eigenvalues also occur in complex conjugate pairs. The objective of the identification procedure is to find estimates of λ_i's, R_i's, A_i's, and n so that the model's output $\hat{y}(t)$ is as close as possible, in SER sense, to the actual system output $y(t)$.

2.2 System Input

For comparison, we will use the same system input and system as in [4], which was identified by the Prony method. Input $u(t)$ is piecewise continuous and characterized by sets of input eigenvalues between points of discontinuity, and for $t \geq 0$ is assumed to be of the general form:

$$u(t)=\sum_{k=1}^{q}\sum_{j=1}^{m} c_{k,j} e^{s_j(t-D_{k-1})}[u_s(t-D_{k-1})-u_s(t-D_k)] \tag{3}$$

which is discontinuous at a finite number of $q+1$ points in time; the u_s terms denote unit step functions. The kth input time interval is characterized by $t\in[D_{k-1}, D_k)$, where $D_0=0$ without loss of generality. A total of $q+1$ time intervals exist for $t \geq 0$, where the $(q+1)$th time interval corresponds to $t \geq D_q$ in which $u(t)=0$. During a given time interval, the input signal is characterized by as many as m eigenvalues s_j, $j=1,2,\cdots,m$. The s_j s are called the input eigenvalues. All values of the input signal parameters are assumed known.

The Laplace transform $U(s)$ for the input signal $u(t)$ of equation (3) is:

$$U(s)=\sum_{k=1}^{q}\sum_{j=1}^{m} c_{k,j}\left[\frac{e^{-sD_{k-1}}-e^{-sD_k+s_j(D_k-D_{k-1})}}{s-s_j}\right] \tag{4}$$

The transform $Y(s)$ of the system output in Fig. 1 is

$$Y(s)=G_i(s)+G(s)U(s) \tag{5}$$

where $G_i(s)$ accounts for initial conditions and is represented by:

$$G_i(s)=\sum_{i=1}^{n}\frac{A_i}{s-\lambda_i} \tag{6}$$

2.3 System Transformation

Since we need to transform the modal form into a controller canonical form or an observer canonical form to estimate transfer functions and initial conditions using AWLS/MFT and other algorithms, we briefly introduce the system transformation.

Assuming distinct poles in the denominator polynomial, and using the well-known Gilbert's diagonal realization scheme [13], a state space realization of $G(s)$ is

$$A_m=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \cdots & & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix},\quad B_m=\begin{bmatrix}R_1\\ R_2\\ \vdots\\ R_n\end{bmatrix},\quad X_{0,m}=\begin{bmatrix}A_1\\ A_2\\ \vdots\\ A_n\end{bmatrix},\quad \begin{matrix}C_m=[1,1,\cdots,1],\\ D_m=R_0\end{matrix} \tag{7}$$

$$\dot{x}(t)=A_m x(t)+B_m u(t) \tag{8}$$

$$y(t)=C_m x(t)+D_m u(t), \text{ with initial condition } x(0)=X_{0,m}$$

Without using similarity transformations, we can directly transform the modal form to a controller canonical form in the case of a SISO system [14]. Let the system transfer function be

$$G(s)=\frac{R_0 s^n+b_1 s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n} \tag{9}$$

The system matrices in controller canonical form $\{A_c, B_c, C_c, D_c\}$ for the above nth order proper transfer function are given by:

$$A_c=\begin{bmatrix}-a_1 & -a_2 & \cdots & -a_n\\ 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix},\quad B_c=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}, \tag{10}$$

$$C_c=[b_1-R_0a_1, b_2-R_0a_2, \cdots, b_n-R_0a_n],\quad D_c=R_0$$

The system matrices in observer canonical form $\{A_o, B_o, C_o, D_o\}$ for the above system are related to (10) by

$$A_o=A_c^T,\ B_o=C_c^T,\ C_o=B_c^T\ D_o=D_c \tag{11}$$

We can apply the same transformation rule to initial response as above. Let the expansion of Laplace transform of zero input response $Y_i(s)$ be

$$Y_i(s)=\sum_{i=1}^{n}\frac{A_i}{s-\lambda_i}=\frac{c_1 s^{n-1}+c_2 s^{n-2}+\cdots+c_{n-1}s+c_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+b_n} \tag{12}$$

where $c_1=\sum_{i=1}^{n}A_i$, $c_2=(-1)\sum_{k=1}^{n}A_k\sum_{i=1, i\neq k}^{n}\lambda_i$, $\cdots$,

$$c_n=(-1)^{n-1}\sum_{k=1}^{n}A_k\prod_{i=1,i\neq k}^{n}\lambda_i, \quad \text{for } n\geq 2 \tag{13}$$

Moreover, A_i can be obtained by partial-fraction expansion theorem in reverse, $A_i=[(s-\lambda_i)Y_i(s)]_{s=\lambda_i}$. Note that if λ_i is complex, the constant A_i is also complex.

The state space representation in controller canonical form for the zero input response, $\{A_{ci}, B_{ci}, C_{ci}\}$, is

$$\dot{x}_i(t)=A_{ci}x_i(t)+B_{ci}\delta(t),\quad X_i(0)=0 \tag{14}$$

$$y_i(t)=C_{ci}x_i(t)$$

where $A_{ci}=A_c, B_{ci}=B_c, C_{ci}=[c_1, c_2, \cdots, c_n]$ (15)

and where $\delta(t)$ is the Dirac delta function.

The initial condition response $Y_i(s)$ is related to the solution of (14) by

$$Y_i(s)=C_{ci}(sI-A_{ci})^{-1}B_{ci}=C_{ci}(sI-A_c)^{-1}B_c \tag{16}$$

Similarly, using system matrices in observer canonical form $\{A_{oi}, B_{oi}, C_{oi}\}$, $Y_i(s)$ can be expressed as

$$Y_i(s)=C_{oi}(sI-A_{oi})^{-1}B_{oi}=C_o(sI-A_o)^{-1}B_{oi} \tag{17}$$

because

$$A_{oi}=A_{ci}^T=A_c^T=A_o,\ B_{oi}=C_{ci}^T,\ C_{oi}=B_{ci}^T=B_c^T=C_o \tag{18}$$

By equating the initial condition response in the s domain, we get

$$\begin{aligned}C_m(sI-A_m)^{-1}X_{0,m}&=C_c(sI-A_c)^{-1}X_{0,c}\\ &=C_{ci}(sI-A_{ci})^{-1}B_{ci}\end{aligned} \tag{19}$$

From either of the above two equations, the IC in controller canonical form, $X_{0,c}$, can be computed. Similarly, the transformed IC in observer canonical form $X_{0,o}$ can be computed from the following relations:

$$\begin{aligned}C_m(sI-A_m)^{-1}X_{0,m}&=C_o(sI-A_o)^{-1}X_{0,o}\\ &=C_{oi}(sI-A_{oi})^{-1}B_{oi}\end{aligned} \tag{20}$$

Substituting (18) into (20), $X_{0,o}$ is obtained by

$$X_{0,o}=B_{oi}=[c_1, c_2, \cdots, c_n]^T \tag{21}$$

where super script T denotes a vector transpose. The objective of identification is to estimate system parameters in (9) and initial conditions in (21). The process of solving for the unknowns involves two main steps:

(1) Initially the model order is assumed to be known *a priori* or it can be determined [16]. Using the AWLS [8] with $y(t)$ and $u(t)$, the transfer function

parameters in equation (9), $\{a_i$'s, R_0, b_i's$\}$, are obtained.

(2) Utilizing the transfer function parameters estimated in the first step, reconstruct a zero state response, $y_u(t)$, and subtract it from the noisy $y(t)$ to obtain a noisy zero input response, $y_i(t)$. With the noisy $y_i(t)$ and noisy output $y(t)$, the initial condition residues in equation (12), the c_i's, can be estimated by several methods which are described in the next section.

3. ESTIMATION OF INITIAL CONDITIONS

This section provides a brief review of several parameter estimation methods that were applied for the estimation of initial conditions in this study.

3.1 Estimation of initial conditions using Gram Matrix

We introduce the Gram matrix method and apply it to the estimation of initial conditions [11]. The zero input response $y_i(t)$ is,

$$y(t) - y_u(t) = Ce^{At}X_0 \tag{22}$$

where $y_u(t)$ denotes the zero state response. Next, we "Square up" the equation and integrate over $[t_i, t_i + \Delta T]$ in order to encompass the known functions.

$$\int_{t_i}^{t_i+\Delta T} e^{A't}C'(y(t) - y_u(t))dt = \int_{t_i}^{t_i+\Delta T} e^{A't}C'Ce^{At}dtX_0, \tag{23}$$

$$i = 0,1,2,\cdots,N-1,\quad \Delta T = T/N,\quad t_i = i\cdot\Delta T.$$

The above equation can be expressed in simple notation as follows:

$$\zeta_i = M_i X_0 \tag{24}$$

where $\zeta_i = \int_{t_i}^{t_i+\Delta T} e^{A't}C'(y(t) - y_u(t))dt$, $M_i = \int_{t_i}^{t_i+\Delta T} e^{A't}C'Ce^{At}dt$

Ce^{At} is a row vector of functions, say $[\xi_1(t), \xi_2(t), \cdots, \xi_n(t)]$ for $t_i \le t \le t_i + \Delta T$. Then each M_i can be expressed by the n vectors:

$$M_i = \begin{bmatrix} \int_{t_i}^{t_i+\Delta T}\xi_1(t)\xi_1(t)dt & \int_{t_i}^{t_i+\Delta T}\xi_1(t)\xi_2(t)dt & \cdots & \int_{t_i}^{t_i+\Delta T}\xi_1(t)\xi_n(t)dt \\ \int_{t_i}^{t_i+\Delta T}\xi_2(t)\xi_1(t)dt & \cdots & \cdots & \vdots \\ \vdots & & & \\ \int_{t_i}^{t_i+\Delta T}\xi_n(t)\xi_1(t)dt & \cdots & \cdots & \int_{t_i}^{t_i+\Delta T}\xi_n(t)\xi_n(t)dt \end{bmatrix} \tag{25}$$

This is the Gram matrix for the n vectors $[\xi_1(t), \xi_2(t), \cdots, \xi_n(t)]$, and the n functions comprising Ce^{At} are linearly dependent if and only if $\det(M_i) = 0$ where M_i is an $n \times n$ real symmetric matrix [14]. This means that knowing $[u(t), y(t)]$ for $t_i \le t \le t_i + \Delta T$ is sufficient to uniquely determine X_0 if $\det\left(\int_{t_i}^{t_i+\Delta T} e^{A't}C'Ce^{At}dt\right) \ne 0$.

For an observation time interval T with sample time interval $T_s = T/N$, equation (24) changes into

$$\sum_{i=1}^{N}\zeta_i = \sum_{i=1}^{N}M_i X_0,\quad i = \{1,2,\cdots,N\} \tag{26}$$

The initial conditions can be estimated by least-squares:

$$\hat{X}_0 = \left(\sum_{i=1}^{N}M_i\right)^{-1}\left(\sum_{i=1}^{N}\zeta_i\right) \tag{27}$$

3.2 Estimation of ICs using Least-Squares

We introduce the least squares method and apply it to the estimation of initial conditions [15]. Provided that the transfer function is accurately estimated by the AWLS, let $y_u(t)$ be a reconstructed zero state response with estimated transfer function parameters. Then the zero input response data can be extracted as follows:

$$y(t) - y_u(t) = Ce^{At}X_0$$

By taking the integral of both sides, we get

$$\int_{t_i}^{t_i+\Delta T}(y(t) - y_u(t))dt = \int_{t_i}^{t_i+\Delta T}Ce^{At}dtX_0 \tag{28}$$

where $i = 0,1,2,\cdots,N-1$, $\Delta T = T/N$, $t_i = i\cdot\Delta T$.

Now define a regression model as

$$Z_i = \phi_i X_0, \tag{29}$$

where $Z_i = \int_{t_i}^{t_i+\Delta T}(y(t) - y_u(t))dt$ and $\varphi_i = \int_{t_i}^{t_i+\Delta T}Ce^{At}dt$

The regression model is

$$\mathrm{Z} = \phi X_0 \tag{30}$$

where $\mathrm{Z} = (Z_1, Z_2, \cdots, Z_N)'$, $\phi = (\phi_1, \phi_2, \cdots, \phi_N)'$. The equation error vector is

$$\varepsilon = \mathrm{Z} - \phi X_0 \tag{31}$$

The least squares estimate of θ is defined as the vector $\hat{\theta}$ that minimizes the loss function $V(\theta) = \frac{1}{2}\varepsilon^T\varepsilon$. Thus, the initial conditions can be estimated by

$$\hat{X}_0 = (\phi^T\phi)^{-1}\phi^T\mathrm{Z} \tag{32}$$

where ϕ is an $N \times n$ regressor matrix, and Z is an $(N \times 1)$ regressand vector. The maximum ΔT is T/n in order to have at least as many regressors as unknown initial conditions.

3.3 Estimation of ICs using Recursive Least-Squares

We introduce the recursive identification method to estimate the initial conditions recursively in time [15]. From a regression model in equation (29), the initial conditions at time t_i can be estimated by

$$\hat{X}_{0,i} = (\phi_i^T\varphi_i)^{-1}\phi_i^T Z_i \tag{33}$$

Using the estimated initial conditions at time t_{i-1}, $\hat{X}_{0,i}$ can be updated as follows:

$$\hat{X}_{0,i} = \hat{X}_{0,i-1} + K(i)\varepsilon(i) \tag{34}$$

where the following definitions apply:

$$\begin{aligned} K(i) &= P(i)\cdot\phi_i^T \\ \varepsilon(i) &= Z_i - \phi_i\cdot\hat{X}_{0,i-1} \\ P(i) &= P(i-1) - \phi_i\cdot P(i-1)/K(i) \\ K(i) &= P(i-1)\cdot\phi_i^T/[1 + \phi_i\cdot P(i-1)\cdot\phi_i^T] \end{aligned} \tag{35}$$

Here the term $\varepsilon(i)$ is interpreted as a prediction error. The algorithm needs initial values $\hat{X}_{0,0}$ and $P(0)$:

$$\hat{X}_{0,0} = \Theta_{n\times 1},\quad P(0) = \rho I_{n\times n} \tag{36}$$

where ρ is a large number.

3.4 Estimation of ICs using Luenberger Observer

This algorithm is suggested in [12] by Professor A. E. Pearson.

Define the initial response

$$z(t) = C_o e^{A_o t}X_{0\rho},\quad 0 < t < T \tag{37}$$

where (A_o, C_o) is any observable state space realization for a given system and $X_{0,o}$ is an initial condition of the observable state space.

Define the reversed I.C. response:

$$z_R(t) = C_o e^{A_o(T-t)} X_{0,o} \\ = C_o x_R(t) \quad (38)$$

where $x_R(t) = e^{A_o(T-t)} X_{0,o}$ satisfies $\dot{x}_R(t) = -A_o x_R(t)$, $x_R(0) = e^{A_o T} X_{0,o}$, and more importantly $x_R(T) = X_{0,o}$.

Hence, we can estimate $x_R(t)$ given $z_R(t)$ via a Luenberger observer as follows:

$$\begin{aligned} \dot{\hat{x}}_R(t) &= -A_o\hat{x}_R(t) + L(z_R(t) - C_o\hat{x}_R(t)) \\ &= -(A_o + LC_o)\hat{x}_R(t) + Lz_R(t) \end{aligned} \quad (39)$$

where L is a column vector for a SISO system.

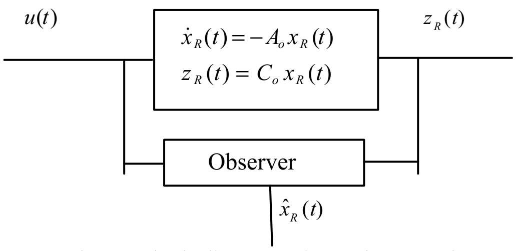


Fig. 2 Block diagram of Luenberger Observer

Let $\tilde{x}(t) = x_R(t) - \hat{x}_R(t)$ to carry out error analysis, then

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{x}_R(t) - \dot{\hat{x}}_R(t) \\ &= -A_o x_R(t) - [-A_o\hat{x}_R(t) + L(C_o x_R(t) - C_o\hat{x}_R(t))] \\ &= -(A_o + LC_o)\tilde{x}(t) \end{aligned} \quad (40)$$

If $-(A_o + LC_o)$ is Hurwitzian with $\tilde{x}(t) = e^{-(A_o+LC_o)T}\tilde{x}(0) \approx 0$, then $x_R(T) - \hat{x}_R(T) \approx 0$ i.e., $\hat{x}_R(T) \approx X_{0,o}$. We design the gain matrix such that $e^{-(A_o+LC_o)T} \approx 0$ and use $\hat{x}_R(T)$ as an estimate of $X_{0,o}$.

4. THE STUDY OF IC ESTIMATION USING LO

In this section, the characteristics of IC estimations for LO method are described using one example model and assuming that the system transfer function parameters are estimated without any error. No noise is added to the output for the simulation in this section.

4.1 Example Model

Let us consider a 4th order system, which is adapted from an article by D. A. Pierre *et al.* [4]. The actual parameters with notations in Fig. 1 are as follows:

$$\lambda_1 = -0.3 + j6.0,\ \lambda_2 = -0.3 - j6.0,\ \lambda_3 = 0,\ \lambda_4 = -1.0;$$
$$A_1 = 0.3 + j0.2,\ A_2 = 0.3 - j0.2,\ A_3 = 0.2,\ A_4 = 0.7; \quad (41)$$
$$R_1 = 1 - j1,\ R_2 = 1 + j1,\ R_3 = 1.2, \text{ and } R_4 = 0.4;$$

The input signal is

$$u(t) = \begin{cases} \sin(0.6\pi t), & 0 \le t < 2 \\ -\sin[2\pi(t-2)], & 2 \le t < 4 \\ 0, & 4 \le t \le 6 \end{cases} \quad (42)$$

which is displayed in Fig. 3. The eigenvalues of the input signal over the first time interval $0 \le t < 2$ are $s_1 = j0.6\pi$ and $s_2 = -j0.6\pi$; over the second time interval $2 \le t < 4$, $s_3 = j2\pi$ and $s_4 = -j2\pi$; and over the third time interval $4 \le t \le 6$, no input eigenvalues apply. The system transfer function and initial response can be obtained by expanding equations (2) and (6):

$$G(s) = \frac{3.6s^3 + 16.76s^2 + 71.064s + 43.308}{s^4 + 1.6s^3 + 36.69s^2 + 36.09s} \quad (43)$$

$$G_i(s) = \frac{1.5s^3 - 0.88s^2 + 30.381s + 7.218}{s^4 + 1.6s^3 + 36.69s^2 + 36.09s} \quad (44)$$

$G(s)$ denotes the system transfer function and $G_i(s)$ denotes the input/output relation of the initial response with the unit impulse function. The actual initial condition in observer canonical form is $X_{0,o} = [1.5,\ -0.88,\ 30.381,\ 7.218]^T$, and the transformed initial condition in controller canonical form is $X_{0,c} = [0.04322; -0.00572; 0.04739; -004450]^T$.

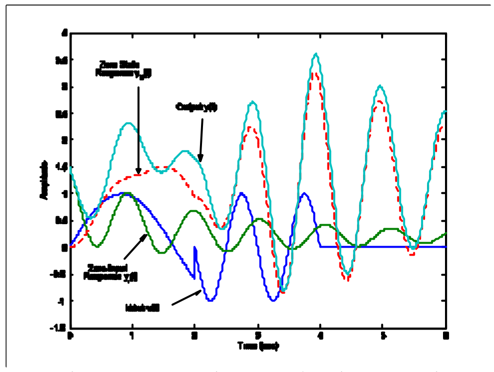


Fig. 3 Input and output for the example model

4.2 ICs Estimation via Luenberger Observer

Define

$$Q = -(A_o + LC_o) \\ = \begin{bmatrix} a_1 - l_1 & -1 & 0 \cdots 0 \\ a_2 - l_2 & 0 & -1 \cdots 0 \\ \vdots & \cdots & \ddots \ \ddots \\ a_n - l_n & 0 & 0 \cdots 0 \end{bmatrix} = \begin{bmatrix} q_1 & -1 & 0 \cdots 0 \\ q_2 & 0 & -1 \cdots 0 \\ \vdots & \cdots & \ddots \\ q_n & 0 & 0 \cdots 0 \end{bmatrix} \quad (45)$$

where $L = [l_1, l_2, \cdots, l_n]^T$ and $q_i = a_i - l_i$, $i = 1,2,\cdots,n$

$$\det(sI - Q) = s^n - q_1 s^{n-1} + q_2 s^{n-2} - \cdots + (-1)^{n-1} q_{n-1} s + (-1)^n q_n \quad (46)$$

Let the desired system matrix A_o^* in observer canonical form be

$$A_o^* = \begin{bmatrix} -a_1^* & 1 & 0 \cdots 0 \\ -a_2^* & 0 & 1 \cdots 0 \\ \vdots & \cdots & \ddots \vdots \\ -a_n^* & 0 & 0 \cdots 0 \end{bmatrix} \quad (47)$$

with characteristic equation

$$\det(sI - A_o^*) = s^n + a_1^* s^{n-1} + \cdots + a_n^* \quad (48)$$

By equating (46) and (48)

$$(-1)^k q_k = a_k^*$$

$$l_k = (-1)^{k+1} a_k^* + a_k \quad \text{for } k = 1,2,\cdots,n \quad (49)$$

The design procedure of gain column vector L is as follows:

1. Choose the desired pole locations or a pole displacement d to move all poles to the left in the complex plane.
2. Transform the diagonal matrix A_m into an observer form A_o^* or write a transfer function with the desired poles to obtain the desired parameters of the denominator $a_1^*, a_2^*, \cdots, a_n^*$
3. Compute the gain vector L using equation (49), where $a_1, a_2, \cdots, a_n$ are estimated in the first phase or are the given denominator parameters of the transfer function.

Using the gain vector L and the state equation (39), $\hat{x}_R(t)$ can be obtained by the MATLAB function:

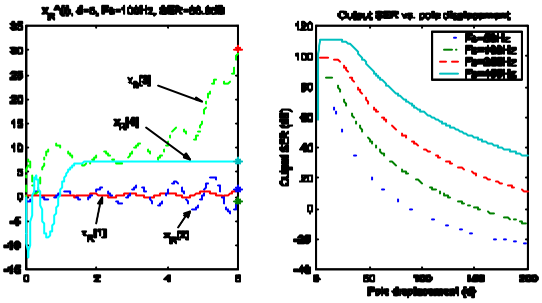
$$\hat{x}_R(t) = LSIM(-(A_o + LC_o), L, I_{n\times n}, \Theta_{n\times 1}, z_R(t), t)$$

One IC estimation example using LO is shown in Fig. 4, where the trace of the estimated $\hat{x}_R(t)$ and the estimated initial condition $\hat{X}_{0,o}$ (the values of $\hat{x}_R(t)$ at t = 6.0 sec designated with * in 10(a)) are illustrated. In this simulation, noise was not added to the initial response which is sampled at 100 Hz, and all system poles are moved to the left by 5, i.e., $d = 5$. Thus the desired pole locations are $\lambda_1^* = -5.3 + j6.0$, $\lambda_2^* = -5.3 - j6.0$, $\lambda_3^* = -5.0$, and $\lambda_4^* = -6.0$. The pole displacement d is defined as

$$d = \mathrm{Re}(\lambda_i^* - \lambda_i) \tag{50}$$

where λ_i^* is the desired pole location and λ_i is the pole of a given system.

In this example, the computed gain vector L is L= [23.2, -174, 1059.08, -1922.7] ^{T}, and the estimated ICs using the Luenberger observer is $X_{0,o}$ = [1.49984, -0.87939, 30.38200, 7.21798]' which produces a model output SER of 86.8 dB. As one can see, the gain l_4 in the column vector L is bigger than any other components. In Fig. 4(a), only in the 4th component of $\hat{x}_R$, $\hat{x}_R[4]$ reaches a steady state after a 1.5 sec transient state but the other states oscillate until they converge to the ICs at time T. As d increases, the transient time of the estimated state $\hat{x}_R(t)$ becomes smaller but its peak values get larger.



(a) Trace of state vector $\hat{x}_R(t)$ (b) Output SER vs. pole displacement

Fig. 4 Estimation of ICs using LO

Fig. 4(b) shows the model output SER, which was computed with the true transfer function and the initial condition $\hat{X}_{0,o}$ estimated by LO for each discrete displacement d from 1 to 200. For the design of the Luenberger observer, all poles were moved to the left by a discrete displacement d, and noise free true $z(t)$ was used for this experiment. The estimated output SER increases as d gets larger up to d = 2 or 3, and keeps the same maximum value to some value of pole movement $d_{\max}$, decreasing after that point. Note that too large a pole displacement causes a big IC estimation error even though it satisfies the criterion, $e^{-(A_o+LC_o)T} \approx 0$. Fig. 4(b) also shows the sampling frequency effects on the estimation of ICs using the LO method. Note that both the model output SER and $d_{\max}$ are proportional to the sampling frequency. The optimum pole displacement d depends on the sampling frequency, but the d should not be too large.

5. SIMULATION RESULTS

In this simulation example, the parameters and initial conditions of a 4th order system are identified under noisy condition: the same noise as in [4], a normally distributed, zero mean white noise with standard deviation of 0.02, which corresponds to 1.15% NSR of output $y(t)$ and to 4.6% NSR of $y_i(t)$, was added to the sampled output prior to identification for comparison. The input and output data were sampled at the same sampling frequency of 50 Hz as in [4]. The actual system parameters are shown in equation (41) in the previous section. The input and noise-free output signals are shown in Fig. 3. The output data $y(t)$ is a combination of the simulated output using LSIM() of MATLAB and the white Gaussian noise sequence generated by RANDN(), i.e.,

$y(t) = LSIM(A_m, B_m, C_m, D_m, u, t, X_{m,0}) + v(t)$ where

$[A_m, B_m, C_m, D_m] = [diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4), (R_1, R_2, R_3, R_4)', (1, 1, 1, 1), 0]$,

$X_{m,0} = [A_1, A_2, A_3, A_4]$, and $v(t) = k * RANDN(301, 1)$ is the additive noise with the scale factor k determining the noise level. If this command does not work with complex parameters in the latest version of MATLAB, use the equations (43) and (44) for generating the output data with real parameters:

$$y(t) = LSIM(G, u, t) + IMPULSE(\ SS(A_i, B_i, C_i, D_i),\ t\ ) + v(t)$$

where

$[A_i, B_i, C_i, D_i] = TF2SS([1.5, -0.88, 30.38, 17.218], [1, 1.6, 36.69, 36.09, 0])$

and $G = TF2SS([3.6, 16.76, 71.064, 43.308], [1, 1.6, 36.69, 36.09, 0])$.

We assumed that the system order, the degree of the system transfer function n, is *a priori* known or is estimated by the technique in [16]. The M user-chosen frequency index is chosen to be 5 which produces the best output SER, so that the highest frequency used in the last regressor, $(M+n)\omega_0$, is 1.5Hz. Note that most of the input energy content is in the frequency range $0 \le f \le 1.5$ Hz, $0 \le \omega \le 9.4$ rad/s. The pole displacement d for the LO method is set to 10, ρ for the RLS is set to 10^{10}, and ΔT for the LS, GRAM, and RLS are all set to the sample interval T_s. Since IC estimates for LO method are obtained only in observer form, all initial conditions for four IC estimation methods are estimated in the observer form for easy comparison. The Signal-to-Error ratio (SER) is defined as

$$SER = 20\log\left(\frac{\|y(t)\|_2}{\|y(t) - \hat{y}(t)\|_2}\right) (dB) \tag{51}$$

which characterizes the estimation performance. Remember that the AWLS was used for the estimation of the transfer function in common, and four methods were used for IC estimation.

Table 1 and 2 show the estimated transfer function parameters and the initial conditions in the noisy output data case. 100 Monte Carlo runs were conducted to mitigate data ambiguity. The estimated ICs by the LS, Gram matrix, and RLS methods are the same as shown in Table 2. Thus, only one model output graph for these three methods is shown in Fig. 5.

For each method, the estimate $\hat{X}_{0,o}[1]$ has the smallest error while the estimate $\hat{X}_{0,o}[4]$ has the largest error among the four initial conditions. The errors of the estimates $\hat{X}_{0,o}[1]$ and $\hat{X}_{0,o}[3]$ for the LO method are smaller than those for the LS, RLS, and GRAM methods but the errors of the estimates

$\hat{X}_{0,o}[2]$ and $\hat{X}_{0,o}[4]$ for the LO method are much larger than those for the other three methods. In the transfer function identification stage, the identified S plane mean eigenvalues are [-0.0043, -0.9815, -0.2992 ± j6.0007]. All the system eigenvalues including the one at s = 0 are accurately identified.

Estimated Parameters of **Transfer Function** (100 Monte Carlo runs, *Fs*=**50Hz**, σ = 0.02)				
True a_i's	a_1 1.6	a_2 36.69	a_3 36.09	a_4 0.0
$\hat{a}_i$'s Mean STD	1.5841 0.1060	36.6919 0.0628	35.5868 3.7616	0.1514 .4658
True b_i's	b_1 3.6	b_2 16.76	b_3 71.064	b_4 43.308
$\hat{b}_i$'s Mean STD	3.5889 0.0295	16.6721 0.4215	70.6921 1.9840	42.3373 6.8531

Table 1 Mean and STD of transfer function parameters

Estimated **Initial Conditions** (100 Monte Carlo runs, *Fs*=**50Hz**, σ = 0.02)				
	$X_{0,o}[1]$ 1.5	$X_{0,o}[2]$ -0.88	$X_{0,o}[3]$ 30.381	$X_{0,o}[4]$ 7.218
LO Mean	1.5029	-0.9750	30.9740	5.3236
STD	0.0146	0.2229	1.3795	4.5696
LS Mean	1.4803	-0.8626	29.6968	7.9595
STD	0.0664	0.1143	2.2982	3.1486

Table 2 Mean and STD of Estimated IC

Fig. 5 shows estimation results by the AWLS and Gram matrix methods in the noisy case. The model output SER of the zero state response is 37.9 dB for both LO and GRAM matrix cases, but SERs of total output, which includes zero state response and zero input response, are 28.72 and 47.08 dB for the LO and Gram matrix method, respectively.

Fig. 6 displays the noisy system output along with the model output identified by the Prony based method. This graph is pasted from [4] for the purpose of visual comparison with our results because no performance measurement is available in [4]. They used the same sample period of $T_s = 0.02$ sec to obtain 301 sampled data and noise with mean zero, standard deviation of 0.02. Assuming the model order n is not given, they used 11 eigenvalues in the model (1) including three dominant ones at s = -0.0293 and s = -0.3126 ± 5.999. However, the system eigenvalue at s = -1 was not readily identified in the remaining model eigenvalues.

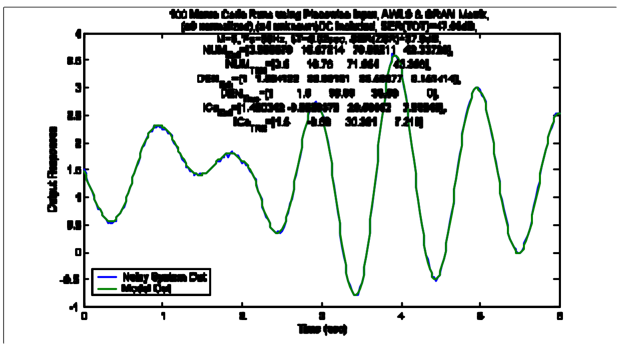


Fig. 5 Model output using AWLS and GRAM matrix

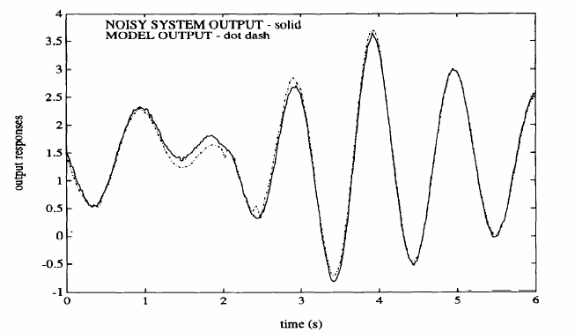


Fig. 6 Model output from Prony method in [4]

The only different condition of this simulation in [4] from ours is the model order, everything else is the same. In other words, we estimated the model order correctly in [16], but they did not specify the model order in the Prony method. Comparing our result in Fig. 5 with the one in Fig. 6, which is identified by the Prony based method under the same simulation conditions, our results show better performance in a sense of error. In particular, our method is very tolerant to noise owing to the benefit of applying the AWLS. In the noise free case, the Prony based method identified poles more accurately than the AWLS, but in the noisy case, the AWLS and one of the three methods, Gram matrix, LS, and RLS, represent better performance than the Prony based method in the output SER aspect. There is one more advantage in our algorithm in that if we increase the sampling rate we could obtain much better results. Even without increasing the sampling rate, we showed that the algorithm we suggested performs better than the Prony method.

6. CONCLUSION

We proposed to use the AWLS/MFT and other estimation methods for the identification of a modal system with nonzero initial conditions. With the benefit of the MFT, we can separate the estimation problem into two phases: the transfer function parameters are estimated in the first phase, and the initial conditions are estimated in the second phase. In addition to the AWLS, we propose the Gram matrix, Luenberger Observer, well-known Least Squares (LS), and Recursive Least Squares (RLS) methods for the estimation of ICs in the second phase. The LO method produces excellent IC estimates in the noise free case [16], but the other three methods show better performance in the noisy case. Finally, we compared our result with the Prony based method in [4]. In the noisy case, the AWLS and one of the three methods - Gram matrix, LS, and RLS- show better performance in the output SER aspect than the Prony based method under the same simulation conditions.

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