On the Dynamics of Multi-Dimensional Lotka-Volterra Equations

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Abstract: In the 3-dimensional cyclic Lotka-Volterra equations, we show the solution on the invariant hyperplane. In addition, we show the existence of the invariant hyperplane by the center manifold theorem under the some conditions. With this result, we can lead the hyperplane of the n-dimensional cyclic Lotka-Volterra equaions. In other section, we study the 3- or 4-dimensional Hamiltonian Lotka-Volterra equations which satisfy the Jacobi identity. We analyze the solution of the Hamiltonian Lotka-Volterra equations with the functions called the split Liapunov functions by [4], [5] since they provide the Liapunov functions for each region separated by the invariant hyperplane. In the cyclic Lotka-Volterra equations, the role of the Liapunov functions is the same in the odd and even dimension. However, in the Hamiltonian Lotka-Volterra equations, we can show the difference of the role of the Liapunov function between the odd and the even dimension by the numerical calculation. In this paper, we regard the invariant hyperplane as the important item to analyze the motion of Lotka-Volterra equations and occur the chaotic orbit. Furtheremore, an example of the asymptoticaly stable and stable solution of the 3-dimensional cyclic Lotka-Volterra equations, 3- and 4-dimensional Hamiltonian equations are shown.

Keywords: Lotka-Volterra equations,invariant hyperplane, center manifold theorem, Liapunov functions, Hamiltonian system

1. Introduction

In this paper, We treat the Lotka-Volterra equations $\dot{x_i} = x_i(b_i + \sum_{j=1}^{n} a_{ij}x_j)$ where $i = 1, \ldots, n$. These equations describe the complex dynamical behavior of systems appearing in the field of biology, ecology, chemistry, physics and economics. There are, however, very few fundamental results on global characteristics of systems that are directly applicable to the control theory. In the following, we restrict ourselves to analysis of the single non-trivial positive fixed point to observe the global dynamic behavior of systems of high dimensions, which gives us the rich information to control the motion of systems including the chaos when it is necessary. We also discuss on the construction of Liapunov functions for a specific class of systems, which makes possible the stabilization of nonlinear systems in systematic ways.

2. Cyclic Lotka-Volterra equations

The cyclic Lotka-Volterra equations describe the connection of prey-predator. Overmore, these equations cover not only prey-predator but also the connection of some species which have the same food. Now we study the 3- and 4-dimensional equations.

2.1. 3-dimensional cyclic Lotka-Volterra equations

The 3-dimensional cyclic Lotka-Volterra equations are the form of,

$$\begin{aligned} \dot{x_1} &= x_1(1 - c_1x_1 - c_2x_2 - c_3x_3), \\ \dot{x_2} &= x_2(1 - c_3x_1 - c_1x_2 - c_2x_3), \\ \dot{x_3} &= x_3(1 - c_2x_1 - c_3x_2 - c_1x_3). \end{aligned} \tag{1}$$

x_1, x_2, x_3 denote the densities, the c_1, c_2, c_3 are describe the effect against each species, which are positive if it enhances and negative if it inhibits the growth. The equations Eqs.(1) have the following fixed points.

$$P_0(0,0,0), \tag{2}$$

$$P_1(\frac{1}{c_1},0,0), \tag{3}$$

$$P_2(0,\frac{1}{c_1},0), \tag{4}$$

$$P_3(0,0,\frac{1}{c_1}), \tag{5}$$

$$P_4(\frac{1}{c_1+c_2+c_3},\frac{1}{c_1+c_2+c_3},\frac{1}{c_1+c_2+c_3}). \tag{6}$$

For analyzing the stability of these points, we investigate the Jacobi matrix of the fixed points P_0, P_1, P_2, P_3 and P_4. Then we use by the way of the stability distinction with the linearlization. Therefore, we also investigate eigenvalues of these Jacobi matrices. The eigenvalue of P_0, λ equals to 1, so P_0 is always unstable. P_1, P_2, P_3 have three same eigenvalues, $\lambda_1 = -1, \lambda_2 = \frac{c_1-c_2}{c_1}, \lambda_3 = \frac{c_1-c_3}{c_1}$. Therefore, the stability of the P_1, P_2, P_3 are denpendent on the coefficient,

- $c_1 < c_2$ and $c_1 < c_3$
  asymptotically stable.
- $c_2 < c_1 < c_3$ or $c_3 < c_1 < c_2$ or $c_2 < c_1$ and $c_3 < c_1$
  unstable.

Then, the eigenvalues of P_4 are $\lambda_1 = -1, \lambda_2 = -\frac{1}{c_1+c_2+c_3}(c_1 - \frac{c_2+c_3}{2} + i\frac{\sqrt{3}}{2}(c_2-c_3)), \lambda_3 = -\frac{1}{c_1+c_2+c_3}(c_1 - \frac{c_2+c_3}{2} - i\frac{\sqrt{3}}{2}(c_2-c_3))$. The condition of the stability of P_4 is following,

- $2c_1 > c_2 + c_3$
  asymptotically stable.
- $2c_1 < c_2 + c_3$
  unstable.

Especially, if it is $2c_1 = c_2+c_3$, the real part of λ_2, λ_3 equal to 0, and they have only the imaginary part. In the nonlinear differential equations, we can not analyze the stability of the system whose eigenvalues of Jacobi matrix have only the imaginary part. Therefore, we discuss the stability of P_4 with the Liapunov function. About Eqs.(1), we can construct the Liapunov function as following [6]

$$V = \frac{x_1 x_2 x_3}{(x_1+x_2+x_3)^3} \geq 0, \quad (7)$$

$$\dot{V} = \frac{x_1 x_2 x_3}{(x_1+x_2+x_3)^4}\left(c_1 - \frac{c_2+c_3}{2}\right)((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2). \quad (8)$$

Depending on the sign of Eq.(8), we can discriminate the stability of P_4. By the Liapunov stability theorem, if $\dot{V} \leq 0$, the system is stable. Therefore, we can find out this fact,

- $2c_1 = c_2 + c_3$
  stable

2.2. The invariant hyperplane

From now, we introduce the invariant hyperplane to analyze the path of Eqs.(1). Generally, when the solution that starts from any initial points on hyper-curved surface belong to it, the hyper-curved surface is called an invariant set. Especially, we call the invariant set the invariant hyperplane if the hyper-curved surface is a hyperplane. For example, the plane of the coordinate is hyperplane. In Lotka-Volterra equations, each x_i means the densities of the population and each x_i must be more than 0. If x_i equals to 0, this fact means that the extermination of the i-th kinds population. The plane of coordinates is a trivial solution, the extermination. Therefore x_i never get over 0 for the future time. In this paper, we investigate the hyperplane except the plane of coordinates.

P_4 is the fixed point which has the different eigenvalues from P_1, P_2, P_3 and does not belong to the coordinates. Therefore we treat the P_4 as the important fixed point for analyzing the cyclic Lotka-Volterra equations.

2.2.1 The first candidate of the invariant hyperplane

When $2c_1 > c_2 + c_3$, P_4 is asymptotically stable, the other fixed points are unstable. Then, we search the invariant hyperplane in the neiborhood of P_4 by the linearlization approximation.

The eigenvalues of P_4's Jacobi matrix has been already calculated. Then, we investigate the eigenvectors of $\lambda_1, \lambda_2, \lambda_3$.

$$y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, y_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix}, y_3 = \begin{bmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}.$$

With these vectors, the divergence of P_4, δx is expressed approximationly

$$\begin{aligned} \delta x &= \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3, \\ &= \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{bmatrix}. \end{aligned} \quad (9)$$

δx is the candidate of the invariant hyperplane which must be expressed with real number. Therefore the plane denoted with $\alpha_1 y_1$ and $\alpha_2 y_2$, or $\alpha_1 y_1$ and $\alpha_3 y_3$ is not the invariant hyperplane. The plane denoted with $\alpha_3 y_3$ and $\alpha_2 y_2$ is may be the invariant hyperplane which has the vertical vector $\vec{u} = (1,1,1)^t$. We assume that asymptotically stable P_4 is on this invariant hyperplane, the equation of the plane is following.

$$x_1 + x_2 + x_3 = \frac{3}{c_1 + c_2 + c_3}. \quad (10)$$

Eq.(10) is the first candidate.

2.2.2 The second candidate

[7]When all the fixed points are unstable, the solution draws among the P_1, P_2 and P_3. From this fact, we think the plane it include these three fixed points

$$x_1 + x_2 + x_3 = \frac{1}{c_1}. \quad (11)$$

Eq.(11) is the second candidate.

2.3. Simulation

From now, we show the result of 3-dimensional cyclic Lotka-Volterra equations, in four terms of "$2c_1 > c_2 + c_3$" , "$2c_1 = c_2 + c_3$" , "$c_1 < c_2 + c_3$ and $c_2 > c_1 > c_3$ or $c_2 < c_1 < c_3$" , "$c_1 < c_2 + c_3$ and $c_1 < c_2$ and $c_1 < c_3$".

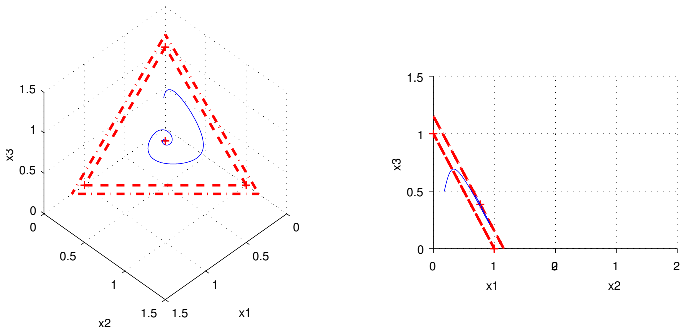

Fig.1 In term of $2c_1 > c_2 + c_3$

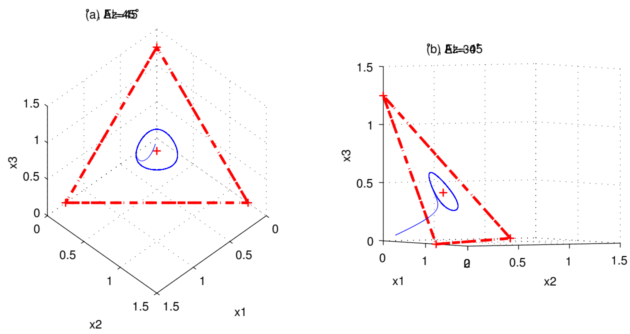

Fig.2 In term of $2c_1 = c_2 + c_3$

- Fig.1 The orbit converged to P_4 on Eq.(10). Under this condition, the invariant hyperplane did not exsit. However, the orbit stayed in the domain $D_0 = \{x | \frac{1}{c_1} \leq x_1 + x_2 + x_3 \leq \frac{3}{c_1+c_2+c_3}\}$ and never went out of D_0.

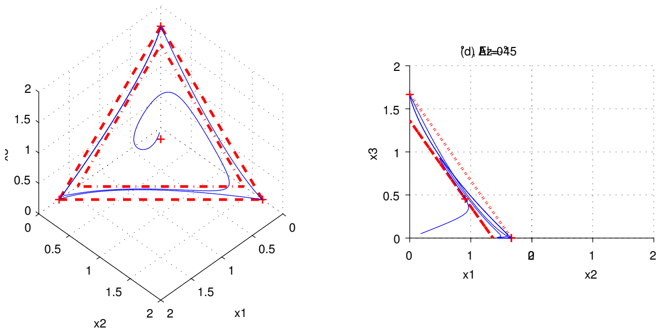

Fig.3 In term of $c_1 < c_2 + c_3$ and $c_2 > c_1 > c_3$ or $c_2 < c_1 < c_3$

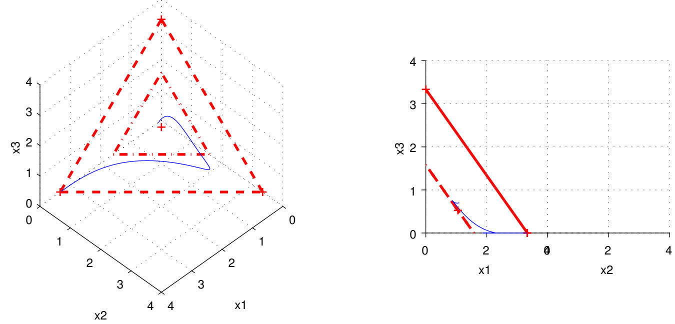

Fig.4 In term of $c_1 < c_2 + c_3$ and $c_1 < c_2$ and $c_1 < c_3$

Namely, we recognized D_0 had the same specific as an invariant set.

- Fig.2 The invariant hyperplane Eq.(10) existed, overmore the orbit drew the limit cycle on the invariant hyperplane.
- Fig.3 The solution drew the orbit between P_1, P_2 and P_3 for the future time.
- Fig.4 The solution converged to P_1. In this case, an invariant hyperplane did not exsit, however we could observe that the orbit never went out of D_0.

2.4. 4-dimensional cyclic Lotka-Volterra equations

The 4-dimensional cyclic Lotka-Volterra equations are the form

$$\begin{aligned} \dot{x_1} &= x_1(1 - c_1x_1 - c_2x_2 - c_3x_3 - c_4x_4), \\ \dot{x_2} &= x_2(1 - c_4x_1 - c_1x_2 - c_2x_3 - c_3x_4), \\ \dot{x_3} &= x_3(1 - c_3x_1 - c_4x_2 - c_1x_3 - c_2x_4), \\ \dot{x_4} &= x_4(1 - c_2x_1 - c_3x_2 - c_4x_3 - c_1x_4).. \end{aligned} \tag{12}$$

The meanings of the variables and coefficients are the same as those of 3-dimensional cyclic Lotka-Volterra equations. Eqs.(12) have the Liapunov function as following,

$$V(\mathbf{x}) = \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^4}, \tag{13}$$

$$\begin{aligned} \dot{V}(\mathbf{x}) = \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^5} & ((3c_1 - c_2 - c_3 - c_4)(x_1^2+x_2^2+x_3^2+x_4^2) \\ & +2(-c_1+c_2-c_3+c_4)(x_1x_2+x_2x_3+x_3x_4+x_4x_1) \\ & +2(-c_1-c_2+3c_3-c_4)(x_1x_3+x_2x_4)). \end{aligned} \tag{14}$$

Through the same argument as 3-dimensions, we can discriminate the stability of Eqs.(12) by the eigenvalues of the Eq.(12)'s Jacobi matrix and by the Liapunov stability approximation which comes from Eq.(14).

3. The invariant hyperplane of the n-dimensional Lotka-Volterra equations

3.1. In 3-dimension

Before n-dimensions, we show the existence of the invariant hyperplane by the center manifold theorem.

<**Proposition 1**>Eqs.(1) have the invariant hyperplane Eq.(10) under the conditon $2c_1 = c_2 + c_3$.

<**Proof**>First, we translate the fixed point P_4 to the origin. After the translation, the Eqs.(1) are

$$\begin{aligned} \dot{x_1} &= -(x_1+\alpha)(c_1x_1 + c_2x_2 + c_3x_3), \\ \dot{x_2} &= -(x_2+\alpha)(c_3x_1 + c_1x_2 + c_2x_3), \\ \dot{x_3} &= -(x_3+\alpha)(c_2x_1 + c_3x_2 + c_1x_3). \end{aligned} \tag{15}$$

Where $\alpha = \frac{1}{c_1+c_2+c_3}$. We set interaction matrix A,

$$A = -\alpha \begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix}. \tag{16}$$

With eigenvectors of A, v_1, v_2, v_3, we make the conversion matrix T afresh.

$$T^{-1} = [v_1 \ v_2 \ v_3]. \tag{17}$$

Then, we convert the variables $\mathbf{x}^t = (x_1 \ x_2 \ x_3)^t$ to $\xi = (u \ v \ w)^t = T\mathbf{x}^t$ and Eqs.(15) are anew written as following,

$$\dot{\xi} = \frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -u + \frac{3}{2}(c_2+c_3)u^2 - \frac{2\sqrt{3}}{3}c_3iuv \\ \frac{3}{2}c_2 + (2 - \frac{\sqrt{3}}{3}i)c_3uv + \lambda_2v \\ \frac{3}{2}c_2 + (2 + \frac{\sqrt{3}}{3}i)c_3uv + \lambda_3v \end{pmatrix}, \tag{18}$$

λ_2, λ_3 are eigenvalues corresponding to eigenvectors v_1, v_2 and they are the imaginary number under the condition $2c_1 = c_2 + c_3$. Next, we assume the center manifold exists which is the form $u = \pi(v, w)$ and use the constance of that motion. Differrentiate u by t,

$$\dot{u} = \frac{\partial\pi}{\partial v}\dot{v} + \frac{\partial\pi}{\partial w}\dot{w}. \tag{19}$$

Then we sustitute Eq.(19) for Eq.(19),

$$\begin{aligned} & -\pi + \frac{3}{2}(c_2+c_3)\pi^2 - \frac{2\sqrt{3}}{3}c_2i\pi v \\ = & \frac{\partial\pi}{\partial v}(B\pi v + \lambda_2 v) + \frac{\partial\pi}{\partial w}(\bar{B}\pi v + \lambda_3 w), \end{aligned} \tag{20}$$

where $B = \frac{3}{2}c_2 + (2 - \frac{\sqrt{3}}{3}i)c_3$, $\bar{B} = \frac{3}{2}c_2 + (2 + \frac{\sqrt{3}}{3}i)c_3$. We also assume that $u = \pi(v, w)$ is the quadratic form of u, v and $\pi(0,0) = 0$. The expression of $u = \pi(v, w)$ is as

$$\pi(v, w) = h_1v + h_2w + h_3v^2 + h_4vw + h_5w^2. \tag{21}$$

We substitute Eq.(21) for Eq.(20) and obtain the idential equation about h_1, h_2, h_3, h_4, h_5. The solution of the idential equation is $h_1 = h_2 = h_3 = h_4 = h_5 = 0$. Accordingly we found the existence of the center manifold which is the form $u = \pi(v, w) = 0$. By T conversion, $u = \frac{1}{3}(x_1 + x_2 + x_3) = 0$. Overmore, re-translation the origin to P_4, $\frac{1}{3}(x_1+x_2+x_3) = 0$ is formed of,

$$x_1 + x_2 + x_3 = \frac{3}{c_1 + c_2 + c_3}, \tag{22}$$

as required. (Q.E.D.)

3.2. In n-dimension

Next subject, we lead the hyperplane of n-dimensional cyclic Lotka-Volterra equations with the same way as 3-dimensions.

<**Proposition 2**>The n-dimensional cyclic Lotka-Volterra equations have the invariant hyperplane which is the form of,

$$\sum_{i=1}^{n} x_i = \frac{n}{\sum_{j=1}^{n} c_j}, \tag{23}$$

consistent with the center manifold.

<**Proof**>The n-dimensional cyclic Lotka-Volterra equations are the form of

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1(1 - c_1x_1 - c_2x_2 - \cdots - c_nx_n) \\ x_2(1 - c_nx_1 - c_1x_2 - \cdots - c_{n-1}x_n) \\ \vdots \\ x_n(1 - c_2x_1 - c_3x_2 - \cdots - c_1x_n) \end{pmatrix}. \tag{24}$$

P is the fixed point which is $P \in R^n_{++} = \mathbf{x} \in R^n_+ : x_i > 0$. We set $\gamma = \sum_{i=1}^{n} c_i$, then $P(\gamma^{-1}, \gamma^{-1}, \cdots, \gamma^{-1})$. After the translation P to the origin, we obtain the interaction matrix A,

$$A = -\gamma^{-1}\begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ & & \vdots & \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}. \tag{25}$$

v_i is the eigenvector corresponding the eigenvalue λ_i. We secure the conversion matrix $T^{-1} = (v_1, v_2, \cdots, v_n)$, then we convert $(y_1, y_2, \cdots, y_n) = T(x_1, x_2, \cdots, x_n)^t$. With T, Eqs.(24) are also convert to,

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}, \tag{26}$$

where δ_i $(i = 1, 2, \ldots, n)$ is the quadratic forms of $y_1, y_2, \ldots, y_n$. We assume that the existence of the center manifold which is $y_1 = \pi(y_2, \ldots, y_n)$, then differentiate y_1 by t,

$$\begin{aligned} \frac{d}{dt}y_1 &= \frac{d}{dt}\sum_{k=2}^{n}\sum_{l=2}^{k} D_{kl}y_ky_l && (27) \\ &= \frac{\partial\pi}{\partial y_2}\dot{y_2} + \cdots + \frac{\partial\pi}{\partial y_n}\dot{y_n} && (28) \\ &= (D_{22}y_2 + D_{23}y_3 + \cdots + D_{2n}y_n)(\lambda_2y_2 + \delta_2) + \cdots && \\ &+ (D_{n2}y_2 + D_{n3}y_3 + \cdots + D_{nn}y_n)(\lambda_ny_n + \delta_n) && (29) \end{aligned}$$

After substitution Eq.(29) for Eq.(26), the solution of the idential equations about D_{kl} is $D_{kl} = 0$, $(k, l = 1, 2, \cdots, n)$. This means that the equation of the invariant hyperplane is $y_1 = x_1 + x_2 + \cdots + x_n = 0$. Then we re-translate P to the origin, and the equation of the hyperplane is

$$\sum_{i=1}^{n} x_i = \frac{n}{\sum_{j=1}^{n} c_j} \tag{30}$$

as required.(Q.E.D.)

4. Hamiltonian Lotka-Volterra equations

The general Lotka-Volterra equations for n-dimensions are the form,

$$\dot{x_i} = x_i(b_i + \sum_{j=1}^{n} a_{ij}x_j), \tag{31}$$

where $i = 1, \cdots$,n. The x_i denote the densities, the b_i are the intrinsic growth or decay rates, and the a_{ij} describe the effect of the jth population upon the ith population, which are positive if it enhances and negative if it inhibits the growth. Then, Eq.(31) is called Hamiltonian system if Eq.(31) can be written as $\dot{\mathbf{x}} = J(\mathbf{x})\nabla H(\mathbf{x})$ where

1. $H(\mathbf{x})$ is a smooth real valued function defined on $\mathbf{R}^n$
2. $J(\mathbf{x})$ is an x-dependent skew-symmetric matrix satisfying the Jacobi-identity

$$\sum_{l=1}^{n}(j_{il}\frac{\partial j_{mk}}{\partial x_l} + j_{kl}\frac{\partial j_{im}}{\partial x_l} + j_{ml}\frac{\partial j_{ki}}{\partial x_l}) = 0, \quad 1 \le i, k, m \le n. \tag{32}$$

Besides these, we set the essence of $J(\mathbf{x})$ to

$$j_{ik} = c_{ik}x_ix_k, \tag{33}$$

and $H(\mathbf{x})$ to a linear function of the form,

$$H = \sum_{l=1}^{n} B_lx_l - x_{n+1}. \tag{34}$$

The equations we can lead after substitution Eq.(33) and (34) for $\dot{\mathbf{x}} = J(\mathbf{x})\nabla H(\mathbf{x})$ are Hamiltonian Lotka-Volterra equations after rescaled the dynamical variables and coefficients. From now, we proove that Eq.(31) have the invariant hyperplane under the one conditions.

<**Proposition 3**>Eqs.(31) admit an invariant hyperplane of the form

$$1 + \sum_{l=1}^{n} B_lx_l = 0, \quad B_l = \frac{a_{ll}}{b_l}, \tag{35}$$

in the following algebraic conditions are satisfied for all $l, j = 1, \ldots, n$

$$B_l(a_{lj} - b_lB_j) = -B_j(a_{jl} - b_jB_l). \tag{36}$$

<**Proof**>Let $\mathbf{x}$ lies on the hyperplane $1 + \sum_{l=1}^{n} = 0$. Then

$$\begin{aligned} \sum_{l=1}^{n} B_l\dot{x_l} &= \sum_{l=1}^{n} B_lb_lx_l + \sum_{l,j=1}^{n} B_la_{lj}x_lx_j, \\ &= (\sum_{l=1}^{n} B_lb_lx_l)(-\sum_{j=1}^{n}) + \sum_{l,j=1}^{n} B_la_{lj}x_lx_j, \\ &= \sum_{l,j=1}^{n} B_l(a_{lj} - b_lB_j)x_lx_j. \end{aligned}$$

This quadratic form is identical to 0 in its coefficient matrix is skew-symmetric, i.e. $B_l(a_{lj} - b_l B_j) = -B_j(a_{jl} - b_j B_l)$ and required.(Q.E.D.)
In the n-dimensional Hamiltonian Lotka-Volterra equations, we can use the function similar to the Liapunov function which are the form,

$$V(x) = (\prod_{l=1}^{n} x_l^{\beta_l})(1 + \sum_{l=1}^{n} B_l x_i), \quad \beta = (b_1 B_1, \dots, b_n B_n)A^{-1}, \tag{37}$$

$$\begin{aligned} \dot{V}(x) &= \prod_{l=1}^{n} B_l x_l^{\beta_l}[(\beta \cdot b)(1 + \sum_{l=1}^{n} B_l x_i) + \sum_{j=1}^{n}(b_j B_j + \sum_{l=1}^{n} \beta_l a_{lj})x_j \\ &+ \sum_{j,l=1}^{n}(B_l a_{lj} + B_l \sum_{k=1}^{n} \beta_k a_{kj})x_j x_l], \qquad (38) \\ &= (\beta \cdot b)V(x). \qquad (39) \end{aligned}$$

$A_{ij} = a_{ij}$ of Eqs.(31). This representation shows that $\dot{V}(x) > 0$ on one side of the invariant hyperplane Eq.(35), $\dot{V}(x) = 0$ on the hyperplane and $\dot{V}(x) < 0$ on the other side. Functions with this property are called the split Liapunov functions by [4], [5], since they provide Liapunov functions for each invariant region separated by the invariant hyperplane. Applying Liapunov's theorem yields that the orbits starting in the interior of the state space go to the boundary, to infinity, or to the invariant hyperplane, which is attracting on the left-hand side and repelling on the right-hand side.

4.1. Numerical analysis

4.1.1 3-dimension

We simulated the 3-dimensional Hamiltonian Lotka-Volterra equations which were $B_l = \frac{a_{ll}}{b_l} = -1$ and A_1, the form of,

$$A_1 = \begin{pmatrix} 2 & -6 & 3 \\ 11 & 3 & 2 \\ -2 & 0 & -1 \end{pmatrix}, \tag{40}$$

in Eq.(35) without lack of generality. A_1 had $(b \cdot \beta) = -\frac{3}{10}$ in Eq.(39). $B_l = \frac{a_{ll}}{b_l} = -1$ meant that the intrinsic growth rate equaled to the rate of the inhibition because of lth population, itself. We instituted three initial points x_{01}, x_{02}, x_{03} which were,

$$x_{01} = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.2 \end{pmatrix}, \ x_{02} = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.3 \end{pmatrix}, \ x_{03} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.5 \end{pmatrix}.$$

Each initial point belonged to each domain separated by an invariant hyperplane $\sum_{i=1}^{3} x_i = 1$. To analyze the orbit easily, we defined the three domains separated by an invariant hyperplane of the systems. The first domain $D_1 = \{x | \sum_{i=1}^{n} x_i < 1\}$ was a set including the origin, the second domain was $D_2 = \{x | \sum_{i=1}^{n} x_i = 1\}$ which was an invariant hyperplane itself, the third domain $D_3 = \{x | \sum_{i=1}^{n} x_i > 1\}$ was a set not including the origin.

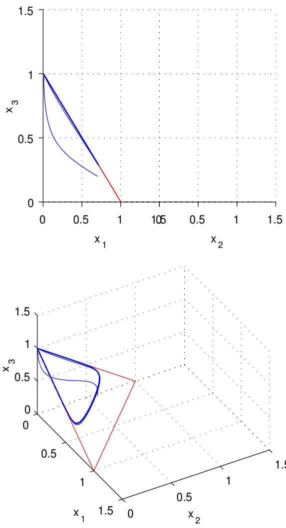

Fig.5 Initial point x_{01}

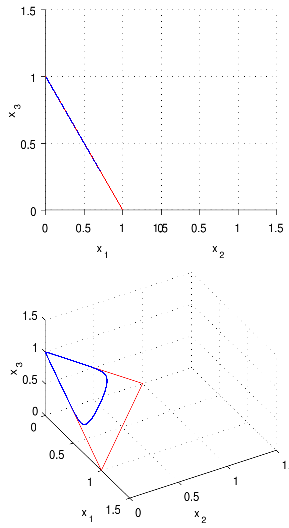

Fig.6 Initial point x_{02}

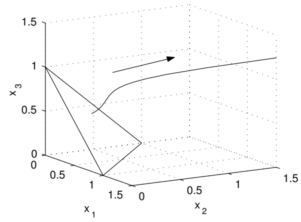

Fig.7 Initial point x_{03}

- Fig.5 This initial point was in D_1. Therefore, Liapunov function Eq.(37) was $V > 0$. Then, because $\dot{V} = -\frac{3}{10}V < 0$, the orbit dropped into the limit cycle on an invariant hyperplane.
- Fig.6 This initial point was in D_2. The Liapunov function Eq.(37) was $V = 0$. Then, $\dot{V} = -\frac{3}{10}V = 0$, therefore the orbit drew the limit cycle on an invariant hyperplane.
- Fig.7 This initial point was in D_3, Liapunov function Eq.(37) is $V < 0$. Therefore, $\dot{V} = -\frac{3}{10}V > 0$. The orbit did not approach the fixed point on invariant hyperplane but diverged.

4.1.2 4-dimension

It was same as 3-dimension, that we simulated under the condition which was $B_l = \frac{a_{ll}}{b_l} = -1$. We constructed the interaction matrix A_2,

$$A_2 = \begin{pmatrix} -1 & -2 & -2 & 0 \\ 0 & -1 & 0 & -4 \\ 2 & 0 & 1 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix}. \tag{41}$$

This system had the fixed point $P_4(0.2, 0.2, 0.2, 0.2)$. Overmore, we used three initial points

$$x_{04} = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.3 \end{pmatrix}, x_{05} = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.4 \\ 0.3 \end{pmatrix}, x_{06} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.3 \\ 0.4 \end{pmatrix}.$$

x_{04} belonged to D_1, x_{05} was on D_2 and x_{06} was in D_3. Now we showed the figures of simulation which was a projection to $x_4 = 0$ to observe the orbit easily.

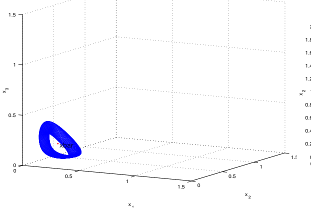

Fig.8 Initial point x_{04}

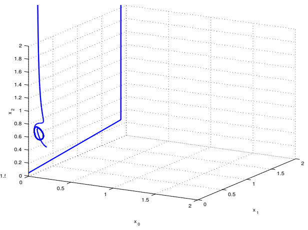

Fig.9 Initial point x_{05}

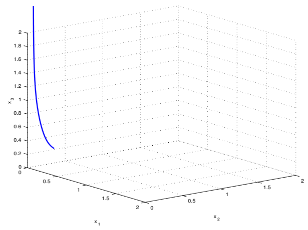

Fig.10 Initial point x_{06}

- Fig.8 Because x_{04} was in D_1, we expected that the orbit drew a cycle. By Fig.8, we could verify the 2-dimensional tori which confined the solution. Then, we investigated the relation of each variable x_1-x_4 and found these result,

$$x_1^2 + x_2^2 = const, \qquad x_3^2 + x_4^2 = const,$$
$$x_1^2 + x_3^2 = const, \qquad x_2^2 + x_4^2 = const. \qquad (42)$$

Therefore, we could recognize the existence of the tori.

- Fig.9 Since x_{05} was on hyperplane, we expected the same result as Fig.8. At first, the solution drew the cycle. However, after a while the solution went out of the cycle and diverged for the future time. For that reason, the hyperplane was in the unstable domain.
- Fig.10 The orbit diverged for the future time.

4.2. Study of Hamiltonian Lotka-Volterra equations

In the above section, we could inspect the difference of the role of an invariant hyperplane between 3-dimension and 4-dimension by an initial point. The Table 1. is the role of an invariant hyperplane in 3- and 4-dimension.

We expand the above, in Hamiltonian Lotka-Volterra equations, the invariant hyperplane's role of even dimension is different from odd dimension. The difference comes from the number of eigenvalues which each dimension has. When the fixed point which has non-zero components is stable, even dimension has the even pair of the imaginary eigenvalus. However, odd dimension has the negative real eigenvalues and the complex eigenvalues that have the negative real number. This describe the difference of the orbit of the solution.

Table 1. The role of the invariant hyperplane

Initial Point	3-dimension	4-dimension
in D_1	The stability of the solution depends on the sign of the split Liapunov function	Unstable at all times
in D_2	The hyperplane works as the invariant hyperplane. The solution draws the limit cycle. The fixed point is on the invariant hyperplane.	The hyperplane does not work as the invariant hyperplane.
in D_3	The stability of the solution depends on the sign of the split Liapunov function	The fixed point exists in D_3. The solution draws two quadratic cycles.

5. Conclusion

We could prove the existence of the invariant hyperplane in the n-dimensional cyclic Lotka-Volterra equation with the results of 3- and 4- dimensions. Moreover, we found the difference of the invariant hyperplane's role between the odd and even dimension in the Hamiltonian Lotka-Volterra equation. With this fact, we could advance the analysis of the n-dimensional Lotka-Volterra equation. We also could find two 2-dimensional tori in the 4-dimensional Hamiltonian Lotka-Volterra equations. In the region between these tori there were at least finitely many periodic orbits, and in general chaotic motion could be expected by Birkhoff's theorem. Moreover, with another conditions, we will be able to occur and control the chaotic orbit, if we can get much deeper information in this systems dynamics. This paper contributed to analyze and control the chaotic motion sufficiently.

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