# On linear output feedback for uncertain nonlinear systems

Ho-Lim Choi, Min-Sung Koo, and Jong-Tae Lim

Dept. of Electrical Engineering
Korea Advanced Institute of Science and Technology
373-1, Kusong-dong, Yusong-gu, Taejon, 305-701, Korea
E-mail: jtlim@stcon.kaist.ac.kr, Fax: +82-42-869-3410, Tel.: +82-42-869-3441

**Abstract:** In this paper, we consider a problem of asymptotic output regulation of a class of uncertain nonlinear systems by output feedback. The system under consideration is in the *Parametric-Pure-Feedback Form*, which does not satisfy the existing conditions such as the triangularity condition or the Lipschitz condition. We propose a linear output feedback controller with a scaling factor, which asymptotically regulates the output of the considered system.

**Keywords:** Asymptotic output regulation, Linear output feedback, Nonlinear systems

## 1. Introduction

The problem of output feedback control of nonlinear systems remains as an active research area. This is a challenging problem mainly because the so-called separation principle generally does not hold for nonlinear systems. Thus, for convenience, one condition which is often assumed in several works is the Lipschitz condition [1]-[2],[4]-[6],[8]. Under the Lipschitz condition, the state estimate error dynamics can be decoupled from the augmented closed-loop system dynamics. Thus, it becomes easier to design an observer and an output feedback controller. This Lipschitz condition is recently relaxed in [7] where only the triangular-type linear growth condition is assumed.

In this paper, the uncertain nonlinear system under consideration is in the *Parametric-Pure-Feedback Form* [3]. This form includes perturbed nonlinear terms which do not satisfy the existing geometrical conditions such as the Lipschitz condition which is mentioned above or the triangularity condition [7],[10]. Thus, the problem of output feedback control of nonlinear systems in the *Parametric-Pure-Feedback Form* seems to be unsolved by the existing methods.

In our proposed method, there are two main steps: In the first step, we define a new state transformation which transforms the considered uncertain nonlinear system into the nonlinear system with uncertainty under the input matching condition. This idea is motivated by [9] where the two-step transformation method is introduced for the treatment of uncertainty. Then, in the second step, for the transformed system we propose a linear output feedback control law with a scaling factor for the asymptotic regulation of the output. In the stability analysis of the closed-loop system, the selection of controller parameters is analytically shown.

## 2. Preliminaries

Consider the following single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= Ax + Bu + \Phi(x,\theta) \\ y &= Cx \end{aligned} \qquad (1)$$

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where $x \in R^n$ is the state, $u \in R$ and $y \in R$ are the input and the output of the system, respectively. The vector $\theta \in D_p \subset R^p$ consists of unknown constants. The system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad (2)$$

The nonlinear term is structured as

$$\Phi(x,\theta) = \begin{bmatrix} \phi_1(x_1, x_2, \theta) \\ \vdots \\ \phi_i(x_1, \cdots, x_{i+1}, \theta) \\ \vdots \\ \phi_n(x_1, \cdots, x_n, \theta) \end{bmatrix} \qquad (3)$$

where $\Phi(0,\theta) = 0$.

The system (1) with the nonlinear term in the form of (3) is called the *Parametric-Pure-Feedback Form* in [3] where the asymptotic stabilizing state feedback control law is developed. This form usually does not satisfy the triangularity condition imposed in [7],[10] (i.e., $|\phi_i(x,\theta)| \le c(|x_1| + \cdots + |x_i|)$, $1 \le i \le n$). Thus, the methods in [7],[10] cannot be directly applied to system (1). Our control objective is to asymptotically regulate the output of the system (1) using a linear output feedback controller. Throughout the paper, the Euclidean 2-norm is used.

**Remark 1:** A large class of nonlinear systems can be represented in the form of (1) via a proper coordinate change. The conditions for the existence of such a coordinate change are addressed in [3],[5].

## 3. System Reformulation

We begin this section with the following assumption imposed on the nonlinear term $\Phi(x,\theta)$:

**Assumption 1:** The function $\phi_i(x_1, \cdots, x_{i+1}, \theta)$, $1 \le i \le n$ is $n-i$ times continuously differentiable with respect to its arguments.

Under Assumption 1, it is obvious that $\dot{\phi}_{i-1}(x,\theta)$ is a continuous function of $(x_1,\cdots,x_{i+1},\theta)$. With this property, we first define

$$\begin{aligned}
\delta_1(x_1,x_2,\theta) &:= \phi_1(x_1,x_2,\theta)\\
\delta_i(x_1,\cdots,x_{i+1},\theta) &:= \phi_i(x_1,\cdots,x_{i+1},\theta)\\
&\quad +\dot{\delta}_{i-1}(x_1,\cdots,x_i,\theta)\\
\delta_n(x_1,\cdots,x_n,\theta,u) &:= \phi_n(x_1,\cdots,x_n,\theta)\\
&\quad +\dot{\delta}_{n-1}(x_1,\cdots,x_n,\theta) \qquad (4)
\end{aligned}$$

where $i=2,\cdots,n-1$. Note that the input $u$ appears in the last function $\delta_n(x,\theta,u)$.
Then, the state transformation $z=T_\theta(x)$ is defined as

$$\begin{aligned}
z_1 &:= x_1\\
z_{i+1} &:= x_2+\delta_i(x_1,\cdots,x_{i+1},\theta) \qquad (5)
\end{aligned}$$

where $i=1,\cdots,n-1$.
With $z=T_\theta(x)$, the system (1) is transformed into the following form:

$$\begin{aligned}
\dot{z} &= Az+Bu+B\delta_n(z,\theta,u)\\
y &= Cz \qquad (6)
\end{aligned}$$

where $\delta_n(z,\theta,u)=\delta_n(x,\theta,u)|_{x=T_\theta^{-1}(z)}$.
From Assumption 1 and the definition of $z=T_\theta(x)$, it is obvious that for a given $D_z$, there exist constants $\gamma\geq 0$ and $\rho\geq 0$ such that

$$\|\delta_n(z,\theta,u)\|\leq\gamma\|z\|+\rho\|u\|,\quad \forall z\in D_z \qquad (7)$$

This linear growth condition is more general than the Lipschitz condition as assumed in [1]-[2],[4]-[6],[8] because we only require the continuity of the function. Obviously, the eq. (7) does not satisfy the triangularity condition imposed in [7],[10].
Also, from the definition of $z=T_\theta(x)$, we note that $\lim_{t\to\infty}\|z\|=0$ guarantees that $\lim_{t\to\infty}\|y\|=0$ and $\|x\|$ is bounded for all $t\geq 0$. Thus, our control goal now is to asymptotically stabilize the system (6) by a linear output feedback controller.

## 4. Linear Output Feedback Control Law

The proposed linear output feedback control law with a scaling factor $\epsilon$ is

$$\begin{aligned}
u &= K(\epsilon)\hat{z} \qquad (8)\\
\dot{\hat{z}} &= A\hat{z}+Bu-L(\epsilon)(y-C\hat{z}) \qquad (9)
\end{aligned}$$

where $K(\epsilon)=[\frac{k_1}{\epsilon^n},\cdots,\frac{k_n}{\epsilon}]$ and $L(\epsilon)=[\frac{l_1}{\epsilon},\cdots,\frac{l_n}{\epsilon^n}]^T$, $\epsilon>0$.
Now, we state the main theorem.
**Theorem 1:** Suppose that $K=[k_1,\cdots,k_n]$ and $L=[l_1,\cdots,l_n]^T$ are selected such that each matrix $A_K:=A+BK$ and $A_L:=A+LC$ is Hurwitz, respectively. Then, there exist positive constants $\rho^*$ and $\epsilon^*$ such that for $0\leq\rho<\rho^*$ and $0<\epsilon<\epsilon^*$, the origin of the system (6) is asymptotically stable by the output feedback control law (8)-(9).

**Proof:** Define $e_i=z_i-\hat{z}_i$, $1\leq i\leq n$. Subtracting the observer dynamics (9) from the system (6), we have the state estimate error dynamics as

$$\dot{e}=A_L(\epsilon)e+B\delta_n(z,\theta,u) \qquad (10)$$

where $A_L(\epsilon):=A+L(\epsilon)C$.
With the controller (8), we have the closed-loop system as

$$\dot{z}=A_K(\epsilon)z+B\delta_n(z,\theta,u)-BK(\epsilon)e \qquad (11)$$

where $A_K(\epsilon):=A+BK(\epsilon)$.
Now, we prove the theorem in three parts.
*Part A:* First, we define a matrix $E_\epsilon:=\text{diag}[1,\epsilon,\cdots,\epsilon^{n-1}]$. Since $A_K$ is Hurwitz, we have a Lyapunov equation $A_K^TP_K+P_KA_K=-I$. Then, using the relation $E_\epsilon^{-1}A_KE_\epsilon=\epsilon A_K(\epsilon)$, we obtain a new Lyapunov equation $A_K(\epsilon)^TP_K(\epsilon)+P_K(\epsilon)A_K(\epsilon)=-\epsilon^{-1}E_\epsilon^2$ where $P_K(\epsilon)=E_\epsilon P_KE_\epsilon$. With this, we set a Lyapunov function $V_c(z)=z^TP_K(\epsilon)z$ for (11). Then, along the trajectory of (11),

$$\begin{aligned}
\dot{V}_c(z) &= -\epsilon^{-1}\|E_\epsilon z\|^2\\
&\quad +2z^TP_K(\epsilon)B\delta_n(z,\theta,u)-2z^TP_K(\epsilon)BK(\epsilon)e\\
&= -\epsilon^{-1}\|E_\epsilon z\|^2\\
&\quad +2z^TE_\epsilon P_KE_\epsilon B\delta_n(z,\theta,u)-2z^TE_\epsilon P_KE_\epsilon BK(\epsilon)e\\
&\leq -\epsilon^{-1}\|E_\epsilon z\|^2+2\|P_K\|\|E_\epsilon z\|\|E_\epsilon B\delta_n(z,\theta,u)\|\\
&\quad +2\|P_K\|\|E_\epsilon z\|\|E_\epsilon BK(\epsilon)e\| \qquad (12)
\end{aligned}$$

Here, we note that

$$\begin{aligned}
\|E_\epsilon B\delta_n(z,\theta,u)\| &\leq \epsilon^{n-1}\|\delta_n(z,\theta,u)\|\\
&\leq \epsilon^{n-1}\gamma\|z\|+\epsilon^{n-1}\rho\|u\| \qquad (13)
\end{aligned}$$

Also, the controller (8) can be expressed as $u=\epsilon^{-n}KE_\epsilon\hat{z}$. Thus, we have

$$\|u\|\leq\epsilon^{-n}\|K\|\|E_\epsilon z\|+\epsilon^{-n}\|K\|\|E_\epsilon e\| \qquad (14)$$

Using a property of $\|z\|\leq\epsilon^{1-n}\|E_\epsilon z\|$, we obtain the following inequality with a simple algebraic manipulation:

$$\|E_\epsilon B\delta_n(z,\theta,u)\| \leq (\gamma+\epsilon^{-1}\rho\|K\|)\|E_\epsilon z\|+\epsilon^{-1}\rho\|K\|\|E_\epsilon e\| \qquad (15)$$

Also, we note that $E_\epsilon BK(\epsilon)e=\epsilon^{-1}BKE_\epsilon e$. Thus,

$$\begin{aligned}
\dot{V}_c(z) &\leq -(\epsilon^{-1}-\sigma_1-2\rho\epsilon^{-1}\sigma_2)\|E_\epsilon z\|^2\\
&\quad +2\epsilon^{-1}(1+\rho)\sigma_2\|E_\epsilon z\|\|E_\epsilon e\| \qquad (16)
\end{aligned}$$

where $\sigma_1=2\gamma\|P_K\|$ and $\sigma_2=\|P_K\|\|K\|$, which are $\epsilon$-independent constants.
*Part B:* The method is similar to Part A. Since $A_L$ is Hurwitz, we have a Lyapunov equation $A_L^TP_L+P_LA_L=-I$. Then, we have the following equalities: $E_\epsilon^{-1}A_LE_\epsilon=\epsilon A_L(\epsilon)$, $A_L^T(\epsilon)P_L(\epsilon)+P_L(\epsilon)A_L(\epsilon)=-\epsilon^{-1}E_\epsilon^2$, and $P_L(\epsilon)=E_\epsilon P_LE_\epsilon$. With this, we set a Lyapunov function $V_o(e)=e^TP_L(\epsilon)e$ for (10). Then, along the trajectory of (10),

$$\begin{aligned}
\dot{V}_o(e) &= -\epsilon^{-1}\|E_\epsilon e\|^2+2e^TE_\epsilon P_LE_\epsilon B\delta_n(z,\theta,u)\\
&\leq -\epsilon^{-1}\|E_\epsilon e\|^2\\
&\quad +2\|P_L\|\|E_\epsilon e\|\|E_\epsilon B\delta_n(z,\theta,u)\| \qquad (17)
\end{aligned}$$

Using (15), we have

$$\begin{aligned}\dot{V}_o(e) &\leq -\epsilon^{-1}(1-2\rho\sigma_4)\|E_\epsilon e\|^2 \\ &\quad +2(\sigma_3+\epsilon^{-1}\rho\sigma_4)\|E_\epsilon e\|\|E_\epsilon z\| \qquad (18)\end{aligned}$$

where $\sigma_3 = \gamma\|P_L\|$ and $\sigma_4 = \|P_L\|\|K\|$, which are $\epsilon$-independent constants.

*Part C:* Now, for the augmented closed-loop system (10)-(11), we set a composite Lyapunov function $V(z,e) = \epsilon V_c(z) + \frac{1}{\sigma_3+\epsilon^{-1}\rho\sigma_4}V_o(e)$. Then, using (16) and (18), we have

$$\begin{aligned}\dot{V}(z,e) &\leq -\epsilon(\epsilon^{-1}-\sigma_1-2\rho\epsilon^{-1}\sigma_2)\|E_\epsilon z\|^2 \\ &\quad +2(1+\rho)\sigma_2\|E_\epsilon z\|\|E_\epsilon e\| \\ &\quad -\frac{\epsilon^{-1}(1-2\rho\sigma_4)}{\sigma_3+\epsilon^{-1}\rho\sigma_4}\|E_\epsilon e\|^2 + 2\|E_\epsilon e\|\|E_\epsilon z\| \\ &= -\begin{bmatrix}\|E_\epsilon e\| \\ \|E_\epsilon z\|\end{bmatrix}^T M \begin{bmatrix}\|E_\epsilon e\| \\ \|E_\epsilon z\|\end{bmatrix} \qquad (19)\end{aligned}$$

where

$$M = \begin{bmatrix} \frac{\epsilon^{-1}(1-2\rho\sigma_4)}{\sigma_3+\epsilon^{-1}\rho\sigma_4} & -(1+\rho)\sigma_2-1 \\ -(1+\rho)\sigma_2-1 & \epsilon(\epsilon^{-1}-\sigma_1-2\rho\epsilon^{-1}\sigma_2) \end{bmatrix}$$

The matrix $M$ is positive definite if and only if $1-2\rho\sigma_4 > 0$ and $\det M(\epsilon) > 0$ where

$$\begin{aligned}\det M(\epsilon) &= \frac{(1-2\rho\sigma_4)(\epsilon^{-1}-\sigma_1-2\rho\epsilon^{-1}\sigma_2)}{\sigma_3+\epsilon^{-1}\rho\sigma_4} \\ &\quad -((1+\rho)\sigma_2+1)^2 \qquad (20)\end{aligned}$$

From (20), the $\det M(\epsilon) > 0$ is satisfied for $0 < \epsilon < \epsilon^*$ where

$$\epsilon^* := \frac{(1-2\rho\sigma_4)(1-2\rho\sigma_2)-\rho\sigma_4((1+\rho)\sigma_2+1)^2}{\sigma_1(1-2\rho\sigma_4)+\sigma_3((1+\rho)\sigma_2+1)^2} \qquad (21)$$

Moreover, from (21), it is obvious that there always exists a constant $\bar{\rho} > 0$ such that $(1-2\rho\sigma_4)(1-2\rho\sigma_2)-\rho\sigma_4((1+\rho)\sigma_2+1)^2 > 0$ for $0 < \rho < \bar{\rho}$. Then, we take $\rho^* := \min\{1/2\sigma_4, \bar{\rho}\}$. Thus, the origin of the system (6) with (8)-(9) is asymptotically stable for $0 \leq \rho < \rho^*$ and $0 < \epsilon < \epsilon^*$. ∎

**Remark 2:** The previous theorem has shown that the asymptotic regulation is achieved for a sufficiently small bound $\rho$ in (7). However, it does not require $\gamma$ to be small because for any finite $\gamma$, there always exists a constant $\epsilon^* > 0$.

## 5. Illustrative Example

Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2 + 0.5\sin(0.01x_2) \\ \dot{x}_2 &= u + \theta x_1^{1/2} \\ y &= x_1 \qquad (22)\end{aligned}$$

where $\theta \in [0,1]$. Obviously, this system does not satisfy the existing conditions [1]-[2],[4]-[8],[10]. Note that $\phi_2(x,\theta) = \theta x_1^{1/2}$ is not Lipschitz at the origin. The first step is to reformulate the system as proposed. By following the proposed method, we obtain

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= u + \delta_2(z,\theta,u) \\ y &= z_1 \qquad (23)\end{aligned}$$

where $z_1 = x_1$, $z_2 = x_2 + \phi_1(x,\theta)$, and $\delta_2(z,\theta,u) = \theta z_1^{1/2} + 0.005\theta z_1^{1/2}\cos(0.01x_2) + 0.005\cos(0.01x_2)u$. Thus, we have the inequality such as $\|\delta_2(z,\theta,u)\| \leq \gamma\|z\| + \rho\|u\|$ where $\gamma = 1.005$ and $\rho = 0.005$. Now, we select $K = [-2.25, -3]$ and $L = [-4, -4]^T$ such that each matrix $A_K$ and $A_L$ is Hurwitz. With this selection, we obtain $\sigma_1 = 2.5139$, $\sigma_2 = 4.7135$, $\sigma_3 = 1.1409$, and $\sigma_4 = 4.2783$. With a simple algebraic manipulation, we obtain that $\rho^* = 0.0063$, which means that the output of the system (22) can be asymptotically regulated by the proposed method. With the obtained information thus far, we compute $\epsilon^* = 0.0212$. We select $\epsilon = 0.02$. Thus, the design of the output feedback controller is completed. The initial values are set as $x_1(0) = 1$ and $x_2(0) = -1$, which is equivalent to $z_1(0) = 1$ and $z_2(0) = -1.005$. From Fig. 1, it is shown that the output is asymptotically regulated and the other state $z_2$ is also regulated. From the definition of $z_2 = x_2 + \phi_1(x,\theta)$, in this case, we also obtain the regulation of state $x_2$.

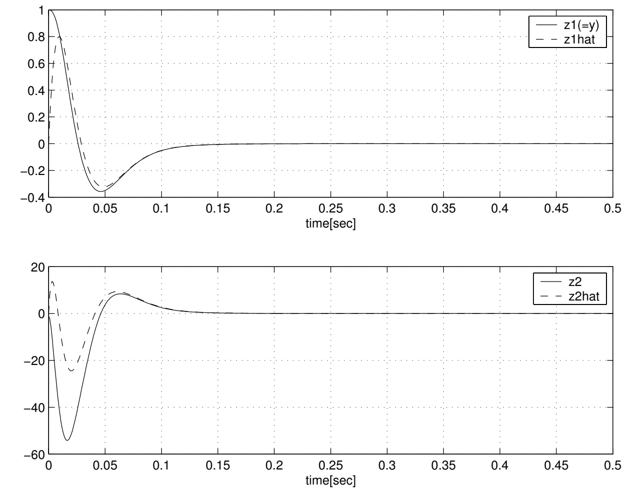


Fig. 1. State trajectories with $(z_1(0), z_2(0)) = (1, -1.005)$ and $(\hat{z}_1(0), \hat{z}_2(0)) = (0, 0)$.

## 6. Conclusions

We have presented the new result on the asymptotic output regulation of uncertain nonlinear systems in the *Parametric-Pure-Feedback Form* by a linear output feedback control scheme. With the new state transformation for system reformulation and the utilization of a scaling factor $\epsilon$, we have analytically shown that the proposed method meets the control goal for the considered uncertain nonlinear systems where the existing methods are not applicable.

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