

Differential Geometric Approach to Sliding Mode Control of Spacecraft Attitude Tracking

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Abstract: Based on the idea that nonlinear PWM controller design can be directly applied to the attitude tracking problem of thruster-controlled spacecraft because it constitutes a sub-class of nonlinear PWM controlled system, nonlinear and output error feedback PWM controlled system is considered to describe the behavior of thruster-controlled spacecraft, and to determine actual thruster on-time which guarantees system stability. A differential geometric approach is utilized to show an asymptotical stability of average PWM system, which finally guarantees the stability of closed loop PWM controlled system. Simulation results show that the motions of PWM controlled system occurs very closely around those of the average model of PWM controlled system.

Keywords: Differential Geometric Method, Sliding Mode Control, Attitude Tracking, Regular Form, Pulse-Width Modulation, Average Model

1. Introduction

In a great variety of practical cases, the control signal in variable structure system is a pulse-modulated signal. A natural name for such a system is referred to as a sampled-data variable structure system. The pulse width modulation (PWM) controlled systems constitute a sub-class of nonlinear sampled-data control system. The sampled output error is translated into a pulse control signal whose pulse width is proportional to the error signal. The PWM controlled systems as variable structure systems are known as robust with respect to parameter variations of system and external perturbation signals [1]. The design approach by using the geometric properties of average PWM controlled system was developed [2][3]. And it is known that the specification of nonlinear PWM controlled system can be made on the basis of the average PWM model, and actual PWM controlled system response shows sliding mode trajectories around sliding manifolds of the average PWM controlled system model [1]. It is very important to determine firing times of thrusters because system response directly depends on it.

In this paper, the problem of sliding mode control for attitude tracking of thruster-controlled spacecraft is considered. Based on the idea that thruster-controlled spacecraft constitutes a sub-class of nonlinear PWM system, nonlinear and output error feedback PWM controlled system is considered to describe the behavior of thruster-controlled spacecraft, and to determine actual thruster on-time which guarantees system stability. The modified Rodrigues parameters (MRP) is used to achieve a minimal parameterization, and *regular form* [4][10] in variable structure systems is considered to characterize spacecraft's dynamic and kinematic equations. A differential geometric approach is utilized to show an asymptotical stability of average PWM system, which finally guarantees the stability of closed loop PWM controlled system.

This paper is organized as follows. In Section 2., a brief review of dynamic and kinematic equations of motion for a three axis stabilized spacecraft is presented. The prop-

erties of output error feedback PWM controlled model and corresponding average model are shown in Section 3. Also, the differential geometric approach in sliding mode control is shown in Section 4. In Section 5, the formulation of thruster-controlled spacecraft system is presented by the nonlinear and output error feedback controlled PWM system, and stability of thruster-controlled spacecraft is shown by using the differential geometric method. Concluding remarks follow in Section 7.

2. Attitude Kinematics and Dynamics

To achieve the minimal parameterization, we use the MRP which is derived by applying stereographic projection [7] of the quaternion, defined as $\beta \equiv [\beta_{13} \ \beta_4]^T$ with $\beta_{13} \equiv [\beta_1 \ \beta_2 \ \beta_3]^T = \hat{n} \sin(\theta/2)$ and $\beta_4 = \cos(\theta/2)$, where \hat{n} is a unit vector corresponding to the axis of rotation and θ is the angle of rotation. The transformation from quaternion to MRP vector σ is given by [7][9]

$$\sigma = \frac{\beta_{13}}{\beta_4 + 1} = \tan\left(\frac{\theta}{4}\right)\hat{n}. \tag{1}$$

The MRP kinematic differential equation in vector form by using spacecraft's body angular velocity (ω), is given by [7][8][9]

$$\dot{\sigma} = \frac{1}{4} \left[(1 - \sigma^2)\mathbf{I}_{3 \times 3} + 2[\sigma \times] + \sigma\sigma^T \right] \omega = B(\sigma)\omega \tag{2}$$

where $\sigma^2 = (\sigma^T \sigma)$ and $\mathbf{I}_{3 \times 3}$ is the 3×3 identity matrix. The cross matrix operator $[\sigma \times]$ is defined by

$$[\sigma \times] \equiv \begin{bmatrix} 0 & -\sigma_3 & \sigma_2 \\ \sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & \sigma_1 & 0 \end{bmatrix}. \tag{3}$$

The inverse transformation of $B(\sigma)$ in explicit vector form is given by [7]

$$B^{-1}(\sigma) = \frac{4}{(1 + \sigma^2)^2} \left[(1 - \sigma^2)\mathbf{I}_{3 \times 3} - 2[\sigma \times] + \sigma\sigma^T \right] \tag{4}$$

Let \mathbf{J} denote the rigid body inertia matrix and \mathbf{u} as the applied torque vector, then Euler's rotational equations of motions for a rigid body are given by

$$\dot{\boldsymbol{\omega}} = f(\boldsymbol{\omega}) + \mathbf{J}^{-1}\mathbf{u} \quad (5)$$

where $f(\boldsymbol{\omega}) = \mathbf{J}^{-1}[\mathbf{J}\boldsymbol{\omega} \times] \boldsymbol{\omega}$.

3. PWM Model and Average Model

Consider an output error feedback PWM controlled system defined in R^n , and described by

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} \quad (6a)$$

$$\mathbf{y} = z(\mathbf{x}) \quad (6b)$$

$$\mathbf{e} = \mathbf{y}_d - \mathbf{y} \quad (6c)$$

$$\mathbf{u} = \text{MPWM}\tau(\mathbf{e}(t_k)) \quad (6d)$$

where f and g are smooth vector fields, z is output function, and \mathbf{y}_d is the desired output value. The control input \mathbf{u} is a discontinuous control vector obtained as the output of a pulse-width-modulator excited by the output error \mathbf{e} . The sampling process is assumed to occur at fixed time interval T , i.e., $t_{k+1} = t_k + T$. M is the maximum allowable input magnitude. The PWM operator $\text{PWM}\tau(\mathbf{e})$ characterizing an on-off switch, is defined as [1]

$$\begin{aligned} \text{PWM}\tau(\mathbf{e}(t_k)) &= \mathbf{u} \\ &= \begin{cases} \text{sign}(\mathbf{e}(t_k)) & \text{for } t_k < t \leq t_k + \tau(\mathbf{x}(t_k))T \\ 0 & \text{for } t_k + \tau(\mathbf{x}(t_k))T < t \leq t_k + T \end{cases} \end{aligned} \quad (7)$$

where t_k is the k -th sampling time and $\tau(\mathbf{e}(t_k))$ is the error dependent duty ratio function defined by

$$\tau(\mathbf{e}(t_k)) = \begin{cases} \alpha|\mathbf{e}(t_k)| & \text{for } |\mathbf{e}(t_k)| \leq 1/\alpha \\ 1 & \text{for } |\mathbf{e}(t_k)| > 1/\alpha \end{cases} \quad (8)$$

where $\alpha > 0$.

It is known that the behavior of the infinite frequency sampled system is described by a nonlinear system which includes a continuous piece-wise smooth control input generated as the output of a memoryless nonlinear function of the saturation type. As the sampling frequency $1/T$ tends to infinity, the nonlinear system in (6), an average model of PWM controlled system, can be described as [1]

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{v} \quad (9a)$$

$$\mathbf{y} = z(\mathbf{x}) \quad (9b)$$

$$\mathbf{e} = \mathbf{y}_d - \mathbf{y} \quad (9c)$$

$$\mathbf{v} = \text{Msat}(\mathbf{e}(t_k), \alpha) \quad (9d)$$

where

$$\text{sat}(\mathbf{e}(t_k), \alpha) = \begin{cases} \alpha\mathbf{e}(t_k) & \text{for } |\mathbf{e}(t_k)| \leq 1/\alpha \\ \text{sign}(\mathbf{e}(t_k)) & \text{for } |\mathbf{e}(t_k)| > 1/\alpha \end{cases}$$

In those regions of the state space where $0 < \tau(\mathbf{x}) < 1$, the state trajectories of the PWM controlled system exhibit a

sliding mode behavior about integral manifolds of the average PWM model. And if the control law $\mathbf{u} = M\text{sign}(\mathbf{e})$ creates a sliding regime locally around the manifold $\mathbf{e} = \mathbf{y}_d - \mathbf{y} = \mathbf{y}_d - z(\mathbf{x})$, then there exists a sufficiently high gain α of the average PWM operator such that the state trajectories of the average PWM system stabilize toward $\mathbf{e} = \mathbf{0}$ [1].

4. Differential Geometric Method in Sliding Mode Control

Consider the nonlinear dynamic system given by

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} \quad (10)$$

where $\mathbf{x} \in R^l$, $\mathbf{u} \in R^l$, and f, g are smooth, local vector fields defined in R^l with $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in R^l$. The sliding surface is the subspace of the state space given by [4]

$$\mathbf{s} \equiv \{\mathbf{x} \in R^l \mid \mathbf{s}(\mathbf{x}) = \mathbf{0}\}. \quad (11)$$

Let $d\mathbf{s}$ be the gradient of $\mathbf{s}(\mathbf{x})$ and \langle, \rangle be the standard scalar product of vectors and co-vectors in their functional relationship. The equivalent control is said to be *well defined* whenever it exists and uniquely determined from the invariance condition [3]. A necessary and sufficient condition for the equivalent control, defined as $\mathbf{u}_{EQ} = -L_f\mathbf{s}/L_g\mathbf{s}$ where $L_a\mathbf{s}$ is the Lie derivative of $\mathbf{s}(\mathbf{x})$ in the direction of $a(\mathbf{x})$, to be well defined is that the *transversality condition* $\langle d\mathbf{s}, g \rangle \neq 0$ is locally satisfied on \mathbf{s} [3]. And if a sliding regime locally exists on \mathbf{s} , then the sliding regime is achieved by a switching logic $\mathbf{u} = k|\mathbf{u}_{EQ}|\text{sign}(\mathbf{s}(\mathbf{x}))$ with $k > 1$ [3].

A system of the form (10) is referred to as a *regular form* in variable structure literature whenever the system is written as [4][10]

$$\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, \mathbf{x}_2) \quad (12a)$$

$$\dot{\mathbf{x}}_2 = f_2(\mathbf{x}_1, \mathbf{x}_2) + g_2(\mathbf{x}_1, \mathbf{x}_2)\mathbf{u} \quad (12b)$$

with $\mathbf{x}_1 \in R^n$, $\mathbf{x}_2 \in R^m$, and $\mathbf{u} \in R^m$. The sliding manifold $\mathbf{s} = \mathbf{0}$ would be specified as

$$\mathbf{s} = \mathbf{x}_2 - \mathbf{s}_0(\mathbf{x}_1) = \mathbf{0} \quad (13)$$

where \mathbf{s}_0 is the solution of equation $\mathbf{s} = \mathbf{0}$ regarding to \mathbf{x}_2 . The equation of motion along the manifold does not depend on the gradient of vector \mathbf{s} and $\mathbf{s}_0(\mathbf{x}_1)$ can be therefore chosen arbitrarily [4]. Under ideal sliding conditions, the equivalent control is obtained using (12b) while (12a) represents the resulting ideal sliding dynamics. The equivalent control is given by [3]

$$\begin{aligned} \mathbf{u}_{EQ}(\mathbf{x}_1) &= \\ &g_2^{-1}(\mathbf{x}_1, \mathbf{s}_0(\mathbf{x}_1)) \left(\left[\frac{\partial \mathbf{s}_0}{\partial \mathbf{x}_1} \right] f_1(\mathbf{x}_1, \mathbf{s}_0(\mathbf{x}_1)) - f_2(\mathbf{x}_1, \mathbf{s}_0(\mathbf{x}_1)) \right). \end{aligned} \quad (14)$$

As a result, we face a design problem for the system (12) with an m -dimensional control rather than $(m+n)$ -dimensional control. A switching surface can be found by designing a nonlinear feedback control law for (12a). When designing control law for (12a), $\mathbf{s}_0(\mathbf{x}_1)$ can be simply treated as the control input to the system.

5. Attitude Tracking of Thruster-Controlled Spacecraft

The nonlinear model for spacecraft motions can be characterized by (2) and (5), and is of *regular form* as (12) with $\mathbf{x}_1 = \boldsymbol{\sigma}$, $\mathbf{x}_2 = \boldsymbol{\omega}$, $f_1 = B$, $f_2 = f(\boldsymbol{\omega})$ and $g_2 = \mathbf{J}^{-1}$, and the thruster-controlled spacecraft system can be represented as nonlinear, output error feedback PWM controlled system described as (6), and described by

$$\dot{\boldsymbol{\omega}} = f(\boldsymbol{\omega}) + \mathbf{J}^{-1}\mathbf{u} \quad (15a)$$

$$\mathbf{y} = \mathbf{s} = \boldsymbol{\omega} - B^{-1}(\boldsymbol{\sigma})d(\boldsymbol{\sigma}) \quad (15b)$$

$$\mathbf{e} = \mathbf{y}_d - \mathbf{y} \quad (15c)$$

$$\mathbf{u} = \text{MPWM}\tau(\mathbf{e}(t_k)) \quad (15d)$$

$$\dot{\boldsymbol{\sigma}} = B(\boldsymbol{\sigma})\boldsymbol{\omega} \quad (15e)$$

$d(\boldsymbol{\sigma})$ determines the desired form of evolution $\boldsymbol{\sigma}$, given by

$$d(\boldsymbol{\sigma}) = \boldsymbol{\Lambda}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_d) \quad (16)$$

where $\boldsymbol{\sigma}_d$ is a desired final value of the attitude parameter, and $\boldsymbol{\Lambda}$ is a diagonal matrix with negative elements. It should be noted that the desired output \mathbf{y}_d becomes zero vector because the sliding manifold is selected as output of the system for feedback purpose. Figure 1 depicts a nonlinear PWM controlled system for thruster controlled spacecraft.

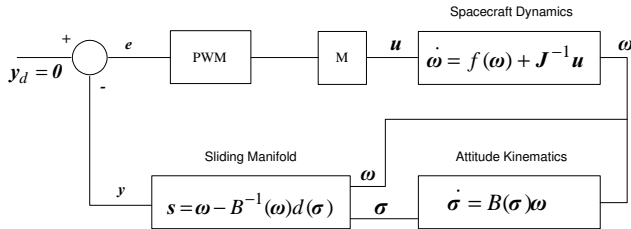


Fig. 1. Sliding Mode Control for Thruster-Controlled Spacecraft System.

It is also known that the closed loop PWM controlled system (6) is asymptotically stable toward the manifold, if and only if the average PWM system (9) is asymptotically stable toward such a manifold [1]. To show an asymptotical stability of closed loop thruster-controlled spacecraft system in (15), a differential geometric approach in Section 4. is used to show an asymptotical stability of average PWM system toward sliding manifold.

Let $\partial/\partial\boldsymbol{\sigma}$ and $\partial/\partial\boldsymbol{\omega}$ be the unit directional vectors spanning the tangent space of R^6 , then the vector fields governing the motions in (10) are given as follows:

$$f = \begin{pmatrix} B(\boldsymbol{\sigma})\boldsymbol{\omega} & \mathbf{0} \\ \mathbf{0} & f(\boldsymbol{\omega}) \end{pmatrix} \begin{pmatrix} \partial/\partial\boldsymbol{\sigma} \\ \partial/\partial\boldsymbol{\omega} \end{pmatrix} \quad (17a)$$

$$g = g_2 = \mathbf{J}^{-1} \frac{\partial}{\partial\boldsymbol{\omega}}. \quad (17b)$$

The *transversality condition* $\langle ds, g \rangle = \mathbf{J}^{-1} \neq \mathbf{0}$ is globally satisfied along sliding manifold, where the gradient to \mathbf{s} is

given by

$$d\mathbf{s} = d\boldsymbol{\omega} + [4\lambda(1 + \boldsymbol{\sigma}^2)^{-1} \{ \mathbf{I}_{3 \times 3} - 2(1 + \boldsymbol{\sigma}^2)^{-1} \boldsymbol{\sigma}\boldsymbol{\sigma}^\top \} - 8\lambda(1 + \boldsymbol{\sigma}^2)^{-2} \{ \boldsymbol{\sigma}\boldsymbol{\sigma}_d^\top - \boldsymbol{\sigma}_d\boldsymbol{\sigma}^\top + [\boldsymbol{\sigma}_d \times] + (\boldsymbol{\sigma}_d^\top \boldsymbol{\sigma}) \mathbf{I}_{3 \times 3} \} + 16\lambda(1 + \boldsymbol{\sigma}^2)^{-3} \{ 1 - \boldsymbol{\sigma}^2 \} \mathbf{I}_{3 \times 3} - 2[\boldsymbol{\sigma} \times] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^\top \} \boldsymbol{\sigma}_d \boldsymbol{\sigma}^\top] d\boldsymbol{\sigma}. \quad (18)$$

The equivalent control torque is obtained using (14) and partial derivative of $\mathbf{s}_0(\boldsymbol{\sigma})$ [6], given by

$$\mathbf{u}_{EQ}(\boldsymbol{\sigma}) = \mathbf{J} \left(\frac{\partial \mathbf{s}_0}{\partial \boldsymbol{\sigma}} B(\boldsymbol{\sigma}) \mathbf{s}_0(\boldsymbol{\sigma}) - \mathbf{J}^{-1} [\mathbf{J} \mathbf{s}_0(\boldsymbol{\sigma}) \times] \mathbf{s}_0(\boldsymbol{\sigma}) \right). \quad (19)$$

The control switching logic, $\mathbf{u} = k|\mathbf{u}_{EQ}|\text{sign}(\mathbf{s}(\mathbf{x}))$ with $k > 1$, is the form of control input in average model (9), and thus the thruster-controlled spacecraft system is asymptotically stable toward the switching regime. Furthermore, if the output is driven to $\mathbf{y} = \mathbf{0}$ and the state trajectories stay on $\mathbf{s} = \mathbf{0}$, then the controlled attitude parameters asymptotically converge to $\boldsymbol{\sigma} = \boldsymbol{\sigma}_d$ and spacecraft's angular velocity $\boldsymbol{\omega}$ also asymptotically tends to zero because the kinematic equation governing the attitude parameter evolution becomes $\dot{\boldsymbol{\sigma}} = \boldsymbol{\Lambda}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_d)$.

6. Simulation

An example of a three-axis spacecraft maneuver is presented and the simultaneous reorientation of all axes is considered here. The following inertia matrix is taken from [5]:

$$\mathbf{J} = \text{diag}[114 \quad 86 \quad 87] \text{ kg}\cdot\text{m}^2$$

The initial conditions for the angular velocity and the modified Rodrigues parameters are given by

$$\boldsymbol{\sigma}(t_0) = [0 \ 0 \ 0]^\top, \quad \text{and}$$

$$\boldsymbol{\omega}(t_0) = [0 \ 0 \ 0]^\top.$$

The desired attitude parameters are given by

$$\boldsymbol{\sigma}_d = [-0.1 \ 0.5 \ 1.0]^\top.$$

The matrix $\boldsymbol{\Lambda}$ is set to $-0.015\mathbf{I}_{3 \times 3} \text{ sec}^{-1}$ and the high gain parameter α is chosen as 50, respectively. The sampling frequency in PWM controlled system is set to 1 sample per 0.25 seconds. Also, the torque of a thruster is set to 1.0 N-m. Figure 2 depicts time trajectories of modified Rodrigues parameters in PWM controlled system, and shows that spacecraft was rotated from the initial position to the desired position. Plots of angular velocity and PWM on-time trajectories are shown in Figure 3 and 4, respectively. The PWM on-time represents firing duration of thrusters with 1.00 N-m for a given sampling period. Also, plots of switching function and angular velocity trajectories are shown in Figure 5 and 6 to compare the responses of PWM controlled system against average model. From Figures 5 and 6, one can see that PWM controlled motions occur very closely around the average model trajectories.

7. Conclusions

The problem of sliding mode control for attitude tracking of thruster-controlled spacecraft was considered. The specification of thruster-controlled spacecraft was made on the basis of nonlinear and output error feedback PWM controlled system by utilizing the MRP and *regular form* in variable structure systems. The thruster firing time which guarantees system stability was given by the PWM controller. The stability of closed loop of thruster-controlled spacecraft system was shown by using the differential geometric method which guarantees asymptotical stability of the average PWM model. Simulation results showed that the motions of PWM controlled system occurs very closely around those of the average model of PWM controlled system.

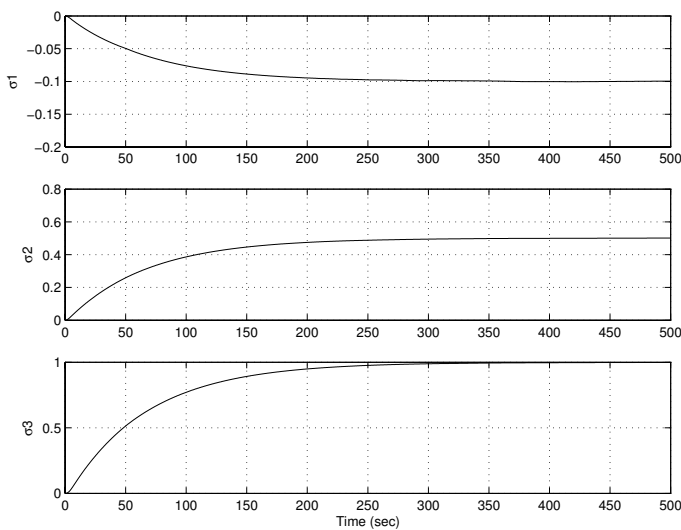


Fig. 2. Plot of PWM Controlled Modified Rodrigues Trajectories.

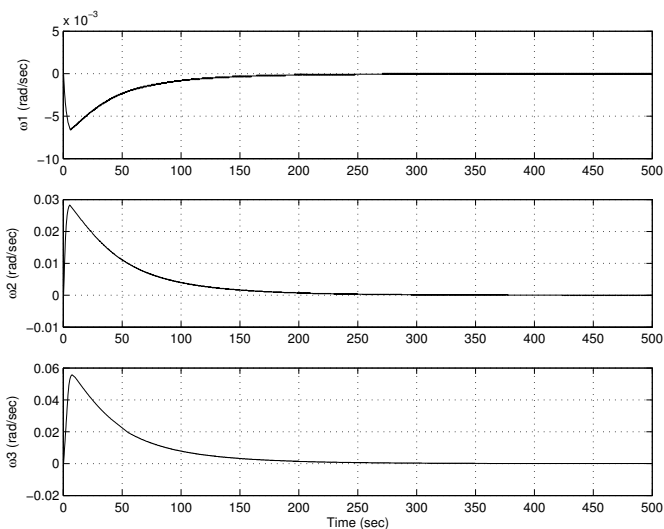


Fig. 3. Plot of PWM Controlled Angular Velocity Trajectories.

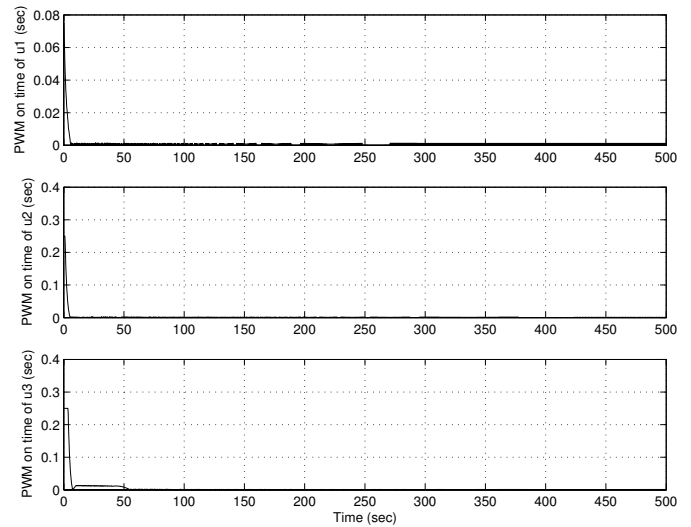


Fig. 4. Plot of PWM On-Time Trajectories.

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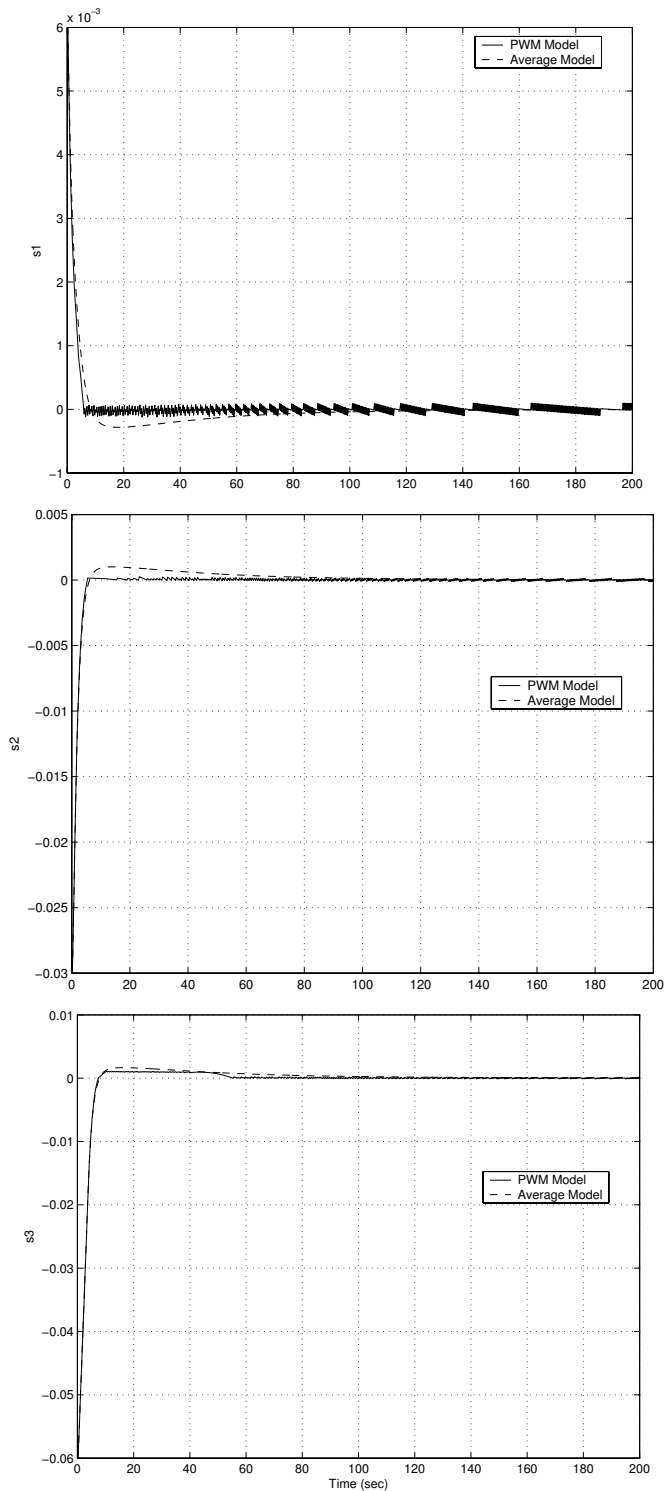


Fig. 5. Plot of Switching Function Trajectories (Average Model vs. PWM Model).

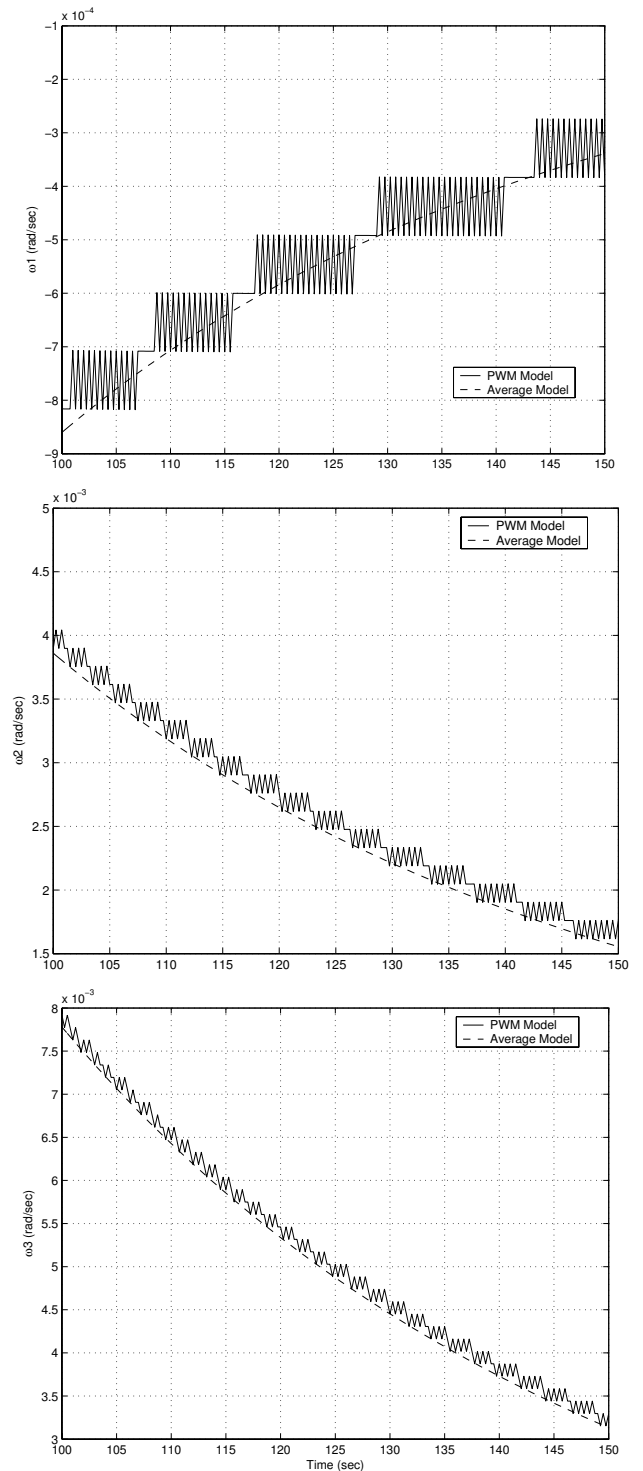


Fig. 6. Plot of Angular Velocity Trajectories (Average Model vs. PWM Model).