Realization and Canonical Representation of Linear Systems through I/O Maps

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Abstract: In this paper, we use the input and output maps and develop simple procedures to obtain realizations for linear continuous-time systems. The procedures developed are numerically efficient and yield explicit formulae for the state space matrices of the realization in terms of the system parameters, notably the system modes. Both cases of the systems with distinct modes and repeated modes are treated. We also present a procedure for converting a realization obtained through the input or output map into the Jordan canonical form. The transformation matrices required to bring the realization into the Jordan canonical form are specified entirely in terms of the system modes.

Keywords: Linear systems, continuous systems, input/output maps, system identification, realization

1. INTRODUCTION

The realization problem requires the determination of an internal state-space description of a system from the knowledge of the input-output (I/O) data. The I/O data is often represented in terms of the transfer function matrix or the impulse response matrix of the system. The problem has been thoroughly investigated for linear time-invariant systems and excellent discussions of the most popular solutions are available in the standard texts on system theory, for example, in [1], [4], and [6]. A relatively recent overview can also be found in the article [9].

An alternative approach utilizing the input maps has been proposed in [3]. These maps are useful in factorizing the impulse response matrix and yield a worthwhile representation of a given transfer function matrix for the purpose of building a balanced state space realization [7]. An important property of the I/O maps, that makes them attractive from the computational point of view, is that the system controllability and observabilty Gramians can be computed from them directly without requiring solutions to the Lyapunov equations. Unfortunately, the results presented in [3] do not produce the state space realization matrices explicitly and, as a result, the dependence of the system parameters, such as the system modes, on these matrices cannot be studied when such a study is desired.

In this paper, we present a computationally efficient formulation of the input map realization approach taken in [3]. We derive a minimal realization for linear continuous time systems when modes are distinct and then extend the results to situations where modes are repeated. We obtain closed form expressions for the controllability and observability Gramians which greatly simplify the computation of the minimal realization. The case of distinct modes is treated in Section 2 and that of repeated modes is considered in Section 3.

In Section 4, we use the results obtained in the previous two sections and provide a method to derive the Jordan canonical form of a realization obtained through the I/O maps. This will be done by breaking down the impulse response matrix of a system in terms of a number of simpler impulse response matrices obtained through a parallel combination of some elementary building blocks. The transformation matrices required to bring the realization into the Jordan canonical form are specified entirely in terms of the system modes. As a result, we will show that a Jordan canonical structure of a system can be obtained efficiently through the use of the I/O maps.

2. REALIZATION WITH DISTINCT MODES

Consider a system whose impulse response matrix has distinct modes, that is,

$$H(t) = \begin{bmatrix} F_1 & F_2 & \dots & F_n \end{bmatrix} I_\phi(t) = I_{\phi r}(t) F^{bT}$$

where

$$\begin{aligned} I_\phi(t) &= \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \otimes I_m, \qquad F^{bT} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}, \\ I_{\phi r}(t) &= \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \dots & e^{\lambda_n t} \end{bmatrix} \otimes I_m. \end{aligned}$$

In this representation, F_i, $i = 1, 2, \dots, n$ are $l \times m$ nonzero constant matrices. It is assumed that the modes have both the geometric as well as algebraic multiplicity of one, that is, $\lambda_i \neq \lambda_j$ whenever $i \neq j$.

The notions of the input and output maps, to be introduced shortly, will rely on the computation of the impulse response matrix derivatives

$$\begin{aligned} H^{(i)}(t) &= \begin{bmatrix} F_1 & F_2 & \dots & F_n \end{bmatrix} \frac{d^i I_\phi(t)}{dt^i} \\ &= \begin{bmatrix} F_1 & F_2 & \dots & F_n \end{bmatrix} \begin{bmatrix} \lambda_1^i e^{\lambda_1 t} \\ \lambda_2^i e^{\lambda_2 t} \\ \vdots \\ \lambda_n^i e^{\lambda_n t} \end{bmatrix} \otimes I_m \\ &= \begin{bmatrix} \lambda_1^i F_1 & \lambda_2^i F_2 & \dots & \lambda_n^i F_n \end{bmatrix} I_\phi(t). \end{aligned}$$

Using $H(t)$ and these derivatives, the so-called coefficient matrix L_V can be formed as

$$V_i(t) = \begin{bmatrix} H(t) \\ H^{(1)}(t) \\ \vdots \\ H^{(i)}(t) \end{bmatrix} = L_V I_\phi(t) = I_{\phi r}(t)\overline{L}_V$$

$$L_V = \begin{bmatrix} F_1 & F_2 & \dots & F_n \\ \lambda_1 F_1 & \lambda_2 F_2 & \dots & \lambda_n F_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^i F_1 & \lambda_2^i F_2 & \dots & \lambda_n^i F_n \end{bmatrix}.$$

Finally, the reduced forms of the coefficient matrix L_V can be utilized to define the input and output maps. The input map is defined as

$$L(t) = e^{At}B = L_a I_\phi(t)$$

where L_a is the row reduced form of L_V, while the output map is defined as

$$R(t) = Ce^{At} = I_{\phi r}(t)R_a$$

where R_a is the column reduced form of L_V.

For a strictly stable system, i.e., when $\Re(\lambda_i) < 0$ for $i = 1, \dots, n$, the input map has the controllability Gramian W_c and the observability Gramian W_o. Using the Kronecker product for matrices, these Gramians can be written down explicitly as

$$\begin{aligned} W_c &= \int_0^\infty L(t)L^*(t)dt = L_a\left[\int_0^\infty I_\phi(t)I_\phi^*(t)\right]L_a^* \\ &= L_a\left[\frac{-1}{\lambda_i+\lambda_j^*}\right]\otimes I_m L_a^*, \\ W_o &= \int_0^\infty R^*(t)R(t)dt = R_a^*\left[\int_0^\infty I_{\phi r}^*(t)I_{\phi r}(t)\right]R_a \\ &= R_a^*\left[\frac{-1}{\lambda_i^*+\lambda_j}\right]\otimes I_m R_a. \end{aligned}$$

Both Gramians are symmetric, and due to the stability assumption, they are also positive definite. These properties allow efficient factorizations of W_c and W_o into their lower and upper diagonal factors using the Cholesky decomposition [5]. Specifically, let T be the normalizing transformation for the input map obtained from the Cholesky decomposition. T is a nonsingular transformation whose computation is known to be numerically stable; it produces the orthonormalized input map

$$\tilde{L}(t) = \tilde{L}_a I_\phi(t) = TL_a I_\phi(t),$$

where $\tilde{L}_a = \left[\ \tilde{L}_{a1}\ \middle|\ \tilde{L}_{a2}\ \middle|\ \dots\ \middle|\ \tilde{L}_{an}\ \right]$, $\tilde{L}_{ai} \in R^{n\times m}$, $i = 1, \dots, n$. A similar procedure can be utilized to normalize the output map as

$$\tilde{R}(t) = I_{\phi r}\tilde{R}_a = I_\phi R_a S,$$

where $\tilde{R}_a = \left[\ \tilde{R}_{a1}\ \middle|\ \tilde{R}_{a2}\ \middle|\ \dots\ \middle|\ \tilde{R}_{an}\ \right]^{bT}$, $\tilde{R}_{ai} \in R^{n\times \ell}$, $i = 1, \dots, n$. Note that T is a normalizing transformation in the sense that the transformed Gramians satisfy the normalized property

$$\begin{aligned} \tilde{W}_c &= \int_0^\infty L(t)L^*(t)dt = L_a\left[\int_0^\infty I_\phi(t)I_\phi^*(t)\right]L_a^* = I_n, \\ \tilde{W}_o &= \int_0^\infty R^*(t)R(t)dt = R_a^*\left[\int_0^\infty I_{\phi r}^*(t)I_{\phi r}(t)\right]R_a = I_n. \end{aligned}$$

We are now in a position to state the main result of this section.

Theorem 1 Let $\Re(\lambda_i) < 0$ for $i = 1, \dots, n$. Then the normalized input map yields the state space realization

$$\begin{aligned} A &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \frac{-\lambda_i}{\lambda_i+\lambda_j^*}\tilde{L}_{ai}\tilde{L}_{aj}^*, \\ B &= \textstyle\sum_{i=1}^n \tilde{L}_{ai}, \\ C &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \frac{-1}{\lambda_i+\lambda_j^*}F_i\tilde{L}_{aj}^*, \end{aligned}$$

while the normalized output map yields the realization

$$\begin{aligned} A &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \frac{-\lambda_j}{\lambda_i^*+\lambda_j}\tilde{R}_{ai}^*\tilde{R}_{aj}, \\ B &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \frac{-1}{\lambda_i^*+\lambda_j}\tilde{R}_{ai}^*F_j, \\ C &= \textstyle\sum_{i=1}^n \tilde{R}_{ai}. \end{aligned}$$

Proof Observe that the derivatives of the normalized input and output maps are given, respectively, by

$$\begin{aligned} \frac{d\tilde{L}(t)}{dt} &= \tilde{L}_a\frac{dI_\phi(t)}{dt} = \tilde{L}_a\begin{bmatrix} \lambda_1 e^{\lambda_1 t} \\ \vdots \\ \lambda_{n_d} e^{\lambda_n t} \end{bmatrix}\otimes I_m \\ \frac{d\tilde{R}(t)}{dt} &= \frac{dI_{\phi r}(t)}{dt}\tilde{R}_a = \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & \dots & \lambda_{n_d} e^{\lambda_n t} \end{bmatrix}\otimes I_m\tilde{R}_a. \end{aligned}$$

Thus, the state matrix is given by

$$\begin{aligned} A &= \frac{d\tilde{L}(t)}{dt}\tilde{L}^*(t)dt = \tilde{L}_a\left[\int_0^\infty \frac{dI_\phi(t)}{dt}I_\phi^*(t)\right]\tilde{L}_a^* \\ &= \tilde{L}_a\left[\frac{-\lambda_i}{\lambda_i+\lambda_j^*}\right]\otimes I_n\tilde{L}_a^* = \left[\frac{-\lambda_i}{\lambda_i+\lambda_j^*}\tilde{L}_{ai}\tilde{L}_{aj}^*\right]_{i,j=1,\dots,n} \\ &= \sum_{i=1}^n\sum_{j=1}^n \frac{-\lambda_i}{\lambda_i+\lambda_j^*}\tilde{L}_{ai}\tilde{L}_{aj}^*. \end{aligned}$$

The input matrix is computed as

$$B = e^{A0}B = \tilde{L}(0) = \tilde{L}_a I_\phi(0) = \sum_{i=1}^n \tilde{L}_{ai},$$

and the output matrix is found as

$$\begin{aligned} C &= C\tilde{W}_c = \int_0^\infty H(t)\tilde{L}^*dt = F\left[\int_0^\infty I_\phi(t)I_\phi^*(t)dt\right] \\ &= F\left[\frac{-1}{\lambda_i+\lambda_j^*}\right]\otimes I_m\tilde{L}_a^* = \sum_{i=1}^n\sum_{j=1}^n \frac{-1}{\lambda_i+\lambda_j^*}F_i\tilde{L}_{aj}^*. \end{aligned}$$

This completes the proof for the first half of the theorem. The proof for the second half of the theorem can be given in a similar way by using the output map instead of the input map. In particular, the matrix B is obtained with the help of the transformed observability Gramian while the matrix C is obtained through the evaluation of the output map at $t = 0$. Details are omitted for brevity. □

3. REALIZATION WITH REPEATED MODES

For repeated modes, we consider the impulse response matrix

$$H(t) = \left[\begin{array}{cccc} F_{i1} & F_{i2} & \dots & F_{in_i} \end{array} \right]_{i=1}^{n} I_\phi(t)$$

where

$$I_\phi(t) = \begin{bmatrix} \mathbf{e}_{\lambda_1} \\ \mathbf{e}_{\lambda_2} \\ \vdots \\ \mathbf{e}_{\lambda_n} \end{bmatrix} \otimes I_m, \qquad \mathbf{e}_{\lambda_i}(\mathbf{t}) = \begin{bmatrix} e^{\lambda_i} \\ te^{\lambda_i} \\ \vdots \\ t^{n_i-1}e_{\lambda_i} \end{bmatrix},$$

the integer n_i is the multiplicity of the i-th mode, and n is the number of distinct modes. Although a mode does not necessarily have the geometric multiplicity of one, its algebraic multiplicity is still assumed to be one. In particular, it is assumed that F_{ij} are nonzero constant matrices of appropriate dimensions and $\lambda_i \neq \lambda_j$ whenever $i \neq j$.

For the repeated modes case, computation of the derivatives of $H(t)$ are more involved. However, using a variant of the well known identity (see [1], p. 203])

$$\frac{d^i(uv)}{dt^i} = \sum_{k=0}^{i} \left(\begin{array}{c} i \\ k \end{array} \right) u^{(k)} v^{(i-k)},$$

namely,

$$\frac{d^i(t^j e^{\lambda t})}{dt^i} = \sum_{k=0}^{\min(i,j)} \left(\begin{array}{c} i \\ k \end{array} \right) \frac{j!}{(j-k)!} t^{j-k} \lambda^{i-k} e^{\lambda t},$$

one can see that the derivatives of the impulse response matrix can be computed explicity as

$$\begin{aligned} H^{(i)}(t) &= \left[\begin{array}{cccc} F_{\ell 1} & F_{\ell 2} & \dots & F_{\ell n_\ell} \end{array} \right]_{\ell=1}^{n} \frac{d^i I_\phi(t)}{dt^i} \\ &= \left[\begin{array}{ccc} \lambda_\ell^i F_{\ell 1} + i\lambda_\ell^{i-1} F_{\ell 2} + \dots & \dots & \lambda_\ell^i F_{\ell n_\ell} \end{array} \right]_{\ell=1}^{n} I_\phi(t). \end{aligned}$$

Once again, the coefficient matrix L_V can be formed using $H(t)$ and its derivatives as

$$V_i(t) = \begin{bmatrix} H(t) \\ H^{(1)}(t) \\ \vdots \\ H^{(i)}(t) \end{bmatrix} = L_V I_\phi(t),$$

where the expression for L_V is given in the Appendix. As in the distinct modes case, we may use the row reduction procedure to obtain a set of linearly independent rows from the coefficient matrix L_V to write the input map in the form

$$L(t) = L_a I_\phi(t).$$

Next, we use the identity

$$\int t^n e^{\lambda t} dt = e^{\lambda t} \sum_{r=0}^{n} (-1)^r \frac{n! t^{n-r}}{(n-r)! \lambda^{r+1}},$$

to evaluate the integral

$$\begin{aligned} W_{ij}(t) &= \int_0^\infty \mathbf{e}_{\lambda_i}(\mathbf{t}) \mathbf{e}^*_{\lambda_i}(\mathbf{t}) \mathbf{dt} \\ &= \int_0^\infty \left[t^{k+\ell} e^{(\lambda_i+\lambda_j^*)t} \right]_{\substack{k=0,1,\dots,n_i-1 \\ \ell=0,1,\dots,n_j-1}} dt \\ &= \left[(-1)^{k+\ell+1} \frac{(k+\ell)!}{(\lambda_i+\lambda_j^*)^{k+\ell+1}} \right]_{\substack{k=0,1,\dots,n_i-1 \\ \ell=0,1,\dots,n_j-1}}. \end{aligned}$$

This result helps us express the controllability Gramian W_c in the closed form

$$W_c = L_a [W_{ij}]_{i,j=1,\dots,n_d} \otimes I_m L_a^*,$$

where W_c is a symmetric matrix.

For a strictly stable system, i.e., when $\Re(\lambda_i) < 0$ for $i = 1, \dots, n$, W_c is a positive definite matrix and can be factorized using the Cholesky decomposition [5]. The Cholesky factors can be used to transform $L(t)$ into the orthonormalized input map $\tilde{L}(t)$ with the transformed controllability Gramian $\tilde{W}_c$ satisfying the normalized property as discussed in the previous section.

We now state the main result of this section.

Theorem 2 Let $\Re(\lambda_i) < 0$ for $i = 1, \dots, n$. Then the normalized input map yields the state space realization

$$\begin{aligned} A &= \textstyle\sum_{i=1}^{n_d} \sum_{j=1}^{n} \tilde{L}_{ai} A_{ij} \tilde{L}_a^*, \\ B &= \textstyle\sum_{i=1}^{n} \tilde{L}_{ai1}, \\ C &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} F_i W_{ij} \tilde{L}_{aj}^*, \end{aligned}$$

where

$$\begin{aligned} A_{ij} &= \left[\begin{array}{l} \frac{(-1)^{k+\ell+1}}{(\lambda_i+\lambda_j^*)^{k+\ell}} (k+\ell-1)! \\ \times \left\{ \frac{\lambda_i}{\lambda_i+\lambda_j^*}(k+\ell) + k \right\} \end{array} \right]_{\substack{k=0,1,\dots,n_i-1 \\ \ell=0,1,\dots,n_j-1}}, \\ W_{ij} &= \left[(-1)^{k+\ell+1} \frac{(k+\ell)!}{(\lambda_i+\lambda_j^*)^{k+\ell+1}} \right]_{\substack{k=0,1,\dots,n_i-1 \\ \ell=0,1,\dots,n_j-1}}. \end{aligned}$$

Proof By a direct calculation we have

$$\frac{d\tilde{L}(t)}{dt} = \tilde{L}_a \begin{bmatrix} \lambda_\ell e^{\lambda_\ell t} \\ (t\lambda_\ell + 1) e^{\lambda_\ell t} \\ \vdots \\ (t^{n_\ell-1}\lambda_\ell + (n_\ell - 1) t^{n_\ell-2}) e^{\lambda_\ell t} \end{bmatrix}_{\ell=1}^{n} \otimes I_m.$$

Thus, the state matrix of the realization is given by

$$\begin{aligned} A &= \frac{d\tilde{L}(t)}{dt} \tilde{L}^*(t) dt = \tilde{L}_a \left[\int_0^\infty \frac{dI_\phi(t)}{dt} I_\phi^*(t) \right] \tilde{L}_a^* \\ &= \tilde{L}_a [A_{ij}]_{i,j=1,\dots,n} \otimes I_m \tilde{L}_a^* = \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{L}_{ai} A_{ij} \tilde{L}_a^*, \end{aligned}$$

where

$$\begin{aligned} A_{ij} &= \left[\int_0^\infty (\lambda_i t^{k+\ell} + k t^{k+\ell-1}) e^{(\lambda_i+\lambda_j^*)t} dt \right]_{\substack{k=0,1,\dots,n_i-1 \\ \ell=0,1,\dots,n_j-1}} \\ &= \left[(-1)^{k+\ell+1} \frac{\lambda_i (k+\ell)!}{(\lambda_i+\lambda_j^*)^{k+\ell+1}} \right. \end{aligned}$$

$$+k(-1)^{k+\ell}\frac{(k+\ell-1)!}{(\lambda_i+\lambda_j^*)^{k+\ell}}\Bigg] \begin{array}{l} k=0,1,\ldots,n_i-1 \\ \ell=0,1,\ldots,n_j-1 \end{array}$$

$$= \left[\frac{(-1)^{k+\ell+1}}{(\lambda_i+\lambda_j^*)^{k+\ell}}(k+\ell-1)! \times \left\{\frac{\lambda_i}{\lambda_i+\lambda_j^*}(k+\ell)+k\right\}\right] \begin{array}{l} k=0,1,\ldots,n_i-1 \\ \ell=0,1,\ldots,n_j-1 \end{array}.$$

The input matrix is given by

$$B=\tilde{L}(0)=\tilde{L}_a I_\phi(0)=\sum_{i=1}^{n}\tilde{L}_{ai1},$$

and the output matrix is given by

$$\begin{aligned} C &= \int_0^\infty H(t)\tilde{L}^*(t)dt = F\left[\int_0^\infty I_\phi(t)I_\phi^*(t)dt\right]\tilde{L}_a^* \\ &= \sum_{i=1}^{n_d}\sum_{j=1}^{n_d} F_i W_{ij}\tilde{L}_{aj}^*. \end{aligned}$$

□

Note that a similar type of realization can be obtained by employing the output map instead of the input map as was done in Theorem 1. Details are omitted here for brevity.

4. JORDAN CANONICAL REPRESENTATION

The results of Theorems 1 and 2 suffer from two shortcomings, namely, in each case
1. The computation is carried out on the entire set of data at once, and
2. In the case of complex modes, the matrices produced for the realization are complex.

In this section, we will show that both shortcomings can be alleviated. To this end, we study the problem as a realization problem for a parallel system of elementary problems, each of which serving as a building block for the original problem. The discussion will be limited to the single-input single output systems and, for the case of repeated modes, only real modes with multiplicity of two will be considered. Considerations for more general cases require further efforts and is not pursued here.

Case 1 Consider a system with the impulse response

$$h(t)=fe^{\lambda t}, t\geq 0,$$

where $\lambda<0$. Using our earlier results we have $I_\phi(t)=e^{\lambda t}$ and $L_V=f$. Thus, the reduced form of L_V is $L_a=1$, and $L(t)=L_aI_\phi(t)=I_\phi(t)=e^{\lambda t}$. A simple calculation gives $W_c=1/(-2\lambda)$ with the Cholesky factor $W_{c\ell}=1/\sqrt{-2\lambda}$. Hence, $\tilde{L}_a=\sqrt{-2\lambda}$. Moreover, $\int_0^\infty \frac{dI_\phi(t)}{dt}I_\phi^*(t)dt=-1/2$. Therefore, the realization of $h(t)$ is given by

$$\Sigma_h=\left[\begin{array}{c|c} \lambda & \sqrt{-2\lambda} \\ \hline \frac{f}{\sqrt{-2\lambda}} & 0 \end{array}\right].$$

Case 2 Consider a system with the impulse response

$$h(t)=f_1e^{\lambda t}+f_2te^{\lambda t}, t\geq 0,$$

where $\Re(\lambda)<0$. This form of $h(t)$ represents the impulse response of a system with a repeated real mode of multiplicity 2. The vector $I_\phi(t)$ and the coefficient matrix L_V of the system are

$$I_\phi(t)=\begin{bmatrix} e^{\lambda t} \\ te^{\lambda^* t}\end{bmatrix}, \qquad L_V=\begin{bmatrix} f_1 & f_2 \\ \lambda f_1+f_2 & \lambda f_2\end{bmatrix}.$$

The reduced form of L_V is $L_a=I_2$ so that $L(t)=I_\phi(t)$. This yields

$$W_c=\frac{1}{-2\lambda}\begin{bmatrix} 1 & \frac{1}{-2\lambda} \\ \frac{1}{-2\lambda} & \frac{1}{2\lambda^2}\end{bmatrix}, \; W_{c\ell}=\frac{1}{\sqrt{-2\lambda}}\begin{bmatrix} 1 & 0 \\ \frac{1}{-2\lambda} & \frac{1}{-2\lambda}\end{bmatrix},$$

where $W_c=W_{c\ell}^*W_{c\ell}$. Therefore, $\tilde{L}_a=W_{c\ell}^{-1}L_a=W_{c\ell}^{-1}$ is given by

$$\tilde{L}_a=\sqrt{-2\lambda}\begin{bmatrix} 1 & 0 \\ -1 & -2\lambda\end{bmatrix}.$$

Finally, by computing the integral

$$\int_0^\infty \frac{dI_\phi(t)}{dt}I_\phi(t)^*dt=\frac{1}{-2\lambda}\begin{bmatrix} \lambda & -1/2 \\ 1/2 & 0\end{bmatrix},$$

we obtain the realization

$$\begin{aligned} A &= \tilde{L}_a\left[\int_0^\infty \frac{dI_\phi(t)}{dt}I_\phi(t)^*dt\right]\tilde{L}_a^*=\begin{bmatrix} \lambda & 0 \\ -2\lambda & \lambda\end{bmatrix}, \\ B &= \tilde{L}_aI_\phi(0)=\sqrt{-2\lambda}\begin{bmatrix} 1 \\ -1\end{bmatrix}, \\ C &= F\left[\int_0^\infty I_\phi(t)I_\phi^*(t)dt\right]\tilde{L}_a^*=\frac{1}{\sqrt{-2\lambda}}\begin{bmatrix} f_1-\frac{f_2}{2\lambda} & -\frac{f_2}{2\lambda}\end{bmatrix}. \end{aligned}$$

Evidently, the realization produced by the algorithm is not in the canonical form. The following lemma provides the transformation of the realization into the canonical form.

Lemma 1 The canonical transformation for the repeated modes case with multiplicity two is given by

$$T_\lambda=\frac{1}{\sqrt{-2\lambda}}\begin{bmatrix} 1+2\lambda & 1 \\ -2\lambda & 0\end{bmatrix}, \; T_\lambda^{-1}=\sqrt{-2\lambda}\begin{bmatrix} 0 & \frac{1}{-2\lambda} \\ 1 & 1+\frac{1}{2\lambda}\end{bmatrix}.$$

Proof Let $T_\lambda=\begin{bmatrix} t_1 & t_2 \\ t_3 & t_4\end{bmatrix}$. Then, a straightforward calculation, based on the similarity transformation

$$T_\lambda\begin{bmatrix} \lambda & 0 \\ -2\lambda & \lambda\end{bmatrix}T_\lambda^{-1}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda\end{bmatrix},$$

shows that the entries of the transformation are constrained as $2t_2^2\lambda=1$, $t_2t_3=-1$, $t_4=0$. This gives the solution $t_2=1/\sqrt{-2\lambda}$, $t_3=-\sqrt{-2\lambda}$, $t_4=0$, with t_1 serving as a free parameter, i.e.,

$$T_\lambda=\frac{1}{\sqrt{-2\lambda}}\begin{bmatrix} t_1 & 1 \\ -2\lambda & 0\end{bmatrix}.$$

We choose t_1 so that the input matrix of the realization is in the canonical form. Since in the new coordinate

$$T_\lambda B=\begin{bmatrix} t_1-1 \\ -2\lambda\end{bmatrix},$$

So, the claim is proved by setting $t_1 = 1+2\lambda$. Note that the matrices of the new realization now take the canonical form

$$\begin{aligned} T_\lambda A T_\lambda^{-1} &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \\ T_\lambda B &= 2\lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\ C T_\lambda^{-1} &= \tfrac{1}{-2\lambda} \begin{bmatrix} f_2 & f_1 + f_2 \end{bmatrix}. \end{aligned}$$

□

Case 3 Consider a system with the impulse response

$$h(t) = f e^{\lambda t} + f^* e^{\lambda^* t}, t \geq 0,$$

where $\Re(\lambda) < 0$. This form of $h(t)$ represents the impulse response of a harmonic oscillator where $\lambda = -(\zeta + j\sqrt{1-\zeta^2})\omega_n$, $f = j\omega_n/(2\sqrt{1-\zeta^2})$, $0 < \zeta \leq 1$, and $\omega_n > 0$. The vector $I_\phi(t)$ and the coefficient matrix L_V of the system are

$$I_\phi(t) = \begin{bmatrix} e^{\lambda t} \\ e^{\lambda^* t} \end{bmatrix}, \qquad L_V = \begin{bmatrix} f & f^* \\ \lambda f & \lambda^* f^* \end{bmatrix}.$$

The reduced form of L_V is $L_a = I_2$ so that $L(t) = I_\phi(t)$. This yields the controllability Gramian

$$W_c = \frac{-1}{2\Re(\lambda)} \begin{bmatrix} 1 & \Re(\lambda)/\lambda \\ \Re(\lambda)/\lambda^* & 1 \end{bmatrix}.$$

Since $W_c > 0$, the Cholesky decomposition of $W_c = W_{c\ell}^* W_{c\ell}$ is found to have the factor

$$W_{c\ell} = \frac{1}{\sqrt{-2\Re(\lambda)}} \begin{bmatrix} 1 & 0 \\ \Re(\lambda)/\lambda^* & \sqrt{1-\Re(\lambda)^2/|\lambda|^2} \end{bmatrix},$$

from which we have $\tilde{L}_a = W_{c\ell}^{-1} L_a = W_{c\ell}^{-1}$, where

$$W_{c\ell}^{-1} = \frac{\sqrt{-2\Re(\lambda)}}{\sqrt{1-\Re(\lambda)^2/|\lambda|^2}} \begin{bmatrix} \sqrt{1-\Re(\lambda)^2/|\lambda|^2} & 0 \\ -\Re(\lambda)/\lambda^* & 1 \end{bmatrix}.$$

Finally, by computing the integral

$$\int_0^\infty \frac{dI_\phi(t)}{dt} I_\phi(t)^* dt = \frac{-1}{2\Re(\lambda)} \begin{bmatrix} \lambda & \Re(\lambda) \\ \Re(\lambda) & \lambda^* \end{bmatrix},$$

we obtain the realization

$$\begin{aligned} A &= \begin{bmatrix} \lambda & 0 \\ (1-\lambda/\lambda^*)\Re(\lambda)/\sqrt{1-\Re(\lambda)^2/|\lambda|^2} & \lambda^* \end{bmatrix}, \\ B &= \sqrt{-2\Re(\lambda)} \begin{bmatrix} 1 \\ (1-\Re(\lambda)/\lambda^*)/\sqrt{1-\Re(\lambda)^2/|\lambda|^2} \end{bmatrix}, \\ C &= \tfrac{f}{\sqrt{-2\Re(\lambda)}} \begin{bmatrix} 1-\Re(\lambda)/\lambda^* & -\sqrt{1-\Re(\lambda)^2/|\lambda|^2} \end{bmatrix}, \end{aligned}$$

where, in obtaining C, we used the fact that $\Re(f) = 0$ which implies $f^* = -f$.

Unfortunately, the matrices produced above are defined over C^n. Therefore, for this realization to be useful, a de-complexification transformation must be performed to bring the entries of these matrices into real forms. The following lemma shows that such a transformation can be characterized entirely in terms of the system modes.

Lemma 2 The de-complexification transformation for a pair of complex conjugate modes is given by

$$\begin{aligned} T_\lambda &= \tfrac{1-j}{2} \begin{bmatrix} j(1-j\Re(\lambda)/\lambda^*) & \sqrt{1-\Re(\lambda)^2/|\lambda|^2} \\ 1+j\Re(\lambda)/\lambda^* & j\sqrt{1-\Re(\lambda)^2/|\lambda|^2} \end{bmatrix}, \\ T_\lambda^{-1} &= \tfrac{1+j}{2\sqrt{1-\Re(\lambda)^2/|\lambda|^2}} \begin{bmatrix} -j\sqrt{1-\Re(\lambda)^2/|\lambda|^2} & \sqrt{1-\Re(\lambda)^2/|\lambda|^2} \\ 1+j\Re(\lambda)/\lambda^* & -j(1-j\Re(\lambda)/\lambda^*) \end{bmatrix}. \end{aligned}$$

Proof We construct the transformation in two steps. In the first step, we bring the system matrix into the diagonal form, and in the next step, we de-complexify the result using the similarity transformation

$$Q \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} Q^{-1} = \begin{bmatrix} \Re(\lambda) & -\Im(\lambda) \\ \Im(\lambda) & Re(\lambda) \end{bmatrix},$$

where

$$Q = \begin{bmatrix} j & -1 \\ 1 & -j \end{bmatrix}, \qquad Q^{-1} = \frac{1}{2} \begin{bmatrix} -j & 1 \\ -1 & j \end{bmatrix}.$$

The transformation required by the first step, namely,

$$P \begin{bmatrix} \lambda & 0 \\ \psi & \lambda^* \end{bmatrix} P^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix}, \ \psi = \frac{(1-\lambda/\lambda^*)\Re(\lambda)}{\sqrt{1-\Re(\lambda)^2/|\lambda|^2}},$$

entails more work due to the redundancy inherited from the triangular structure of the system matrix. We use this redundancy to our advantage for the purpose of de-complexification. To this end, we partition P as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix},$$

and observe that the constraints

$$p_1 \neq 0, \ p_3 \neq 0, \ p_4 \neq 0, \ p_2 = 0, \ p_3(\lambda - \lambda^*) + p_4 \psi = 0,$$

are to be imposed on the entries of P, the last of which gives

$$p_4 = -\left(\frac{\lambda - \lambda^*}{\psi}\right) p_3.$$

Thus, the free parameters p_1 and p_3 are to be chosen so that the transformation $T_\lambda = QP$ renders the original realization into the de-complexified form. This transformation is given by

$$\begin{aligned} \begin{bmatrix} \Re(\lambda) & -\Im(\lambda) \\ \Im(\lambda) & \Re(\lambda) \end{bmatrix} &= Q \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} Q^{-1} \\ &= QP \begin{bmatrix} \lambda & 0 \\ \psi & \lambda^* \end{bmatrix} P^{-1} Q^{-1} \\ &= T_\lambda \begin{bmatrix} \lambda & 0 \\ \psi & \lambda^* \end{bmatrix} T_\lambda^{-1}, \end{aligned}$$

where

$$T_\lambda = \begin{bmatrix} jp_1 - p_3 & \left(\frac{\lambda-\lambda^*}{\psi}\right) p_3 \\ p_1 - jp_3 & j\left(\frac{\lambda-\lambda^*}{\psi}\right) p_3 \end{bmatrix}.$$

Since the system matrix is already de-complexified without further restrictions on p_1 and p_3, we select these parameters so that the remaining matrices, namely the input and output matrices, are de-complexified as well. Evidently, the de-complexification of one of these two matrices will suffice, so we proceed arbitrarily with the decomplexification of the input matrix

$$B = \sqrt{-2\Re(\lambda)} \begin{bmatrix} 1 \\ \beta \end{bmatrix}, \qquad \beta = \frac{1 - \Re(\lambda)/\lambda^*}{\sqrt{1 - \Re(\lambda)^2/|\lambda|^2}},$$

which has the representation

$$T_\lambda B = \sqrt{-2\Re(\lambda)} \begin{bmatrix} jp_1 + (\alpha - 1)p_3 \\ p_1 + j(\alpha - 1)p_3 \end{bmatrix}, \; \alpha = \left(\frac{\lambda - \lambda^*}{\psi} \right) \beta$$

in the new coordinate system. Further restriction can be imposed by seeking a canonical structure which, in turn, translates into the conditions $jp_1 + (\alpha - 1)p_3 = 1$, $p_1 + j(\alpha - 1)p_3 = 1$. Solving these equations yields the solution $p_1 = \frac{1-j}{2}$, $p_3 = \frac{1-j}{2}\left(\frac{1}{\alpha-1}\right)$, and the resulting transformation is

$$\begin{aligned} P &= \frac{1-j}{2\lambda^*} \begin{bmatrix} \lambda^* & 0 \\ -\Re(\lambda) & -\lambda^* \sqrt{1 - \Re(\lambda)^2/|\lambda|^2} \end{bmatrix}, \\ P^{-1} &= \frac{2}{1-j} \begin{bmatrix} 1 & 0 \\ -\Re(\lambda)/(\lambda^* \sqrt{1 - \Re(\lambda)^2/|\lambda|^2}) & -1/\sqrt{1 - \Re(\lambda)^2/|\lambda|^2} \end{bmatrix}, \end{aligned}$$

which was obtained upon back substitution. Therefore, the overall transformation, T_λ, can be obtained from $T_\lambda = QP$, and its inverse, T_λ^{-1}, from $T_\lambda^{-1} = P^{-1}Q^{-1}$. It is straightforward to verify that the results of this computation agree with the ones claimed by the lemma. Finally, we observe that, under this transformation, the matrices of the realization take the canonical form

$$\begin{aligned} T_\lambda A T_\lambda^{-1} &= \begin{bmatrix} \Re(\lambda) & -\Im(\lambda) \\ \Im(\lambda) & \Re(\lambda) \end{bmatrix}, \\ T_\lambda B &= \sqrt{-2\Re(\lambda)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C T_\lambda^{-1} &= \tfrac{jf}{\sqrt{-2\Re(\lambda)}} \begin{bmatrix} -1 & 1 \end{bmatrix}. \end{aligned}$$

Note that the transformed output matrix has real entries since f is purely imaginary. □

5. APPENDIX

The coefficient matrix for the repeated modes case is given by (see the equations leading to Theorem 2)

$$L_V = \begin{bmatrix} \begin{bmatrix} F_{\ell 1} & F_{\ell 2} & \dots & F_{\ell n_\ell} \end{bmatrix}_{\ell=1}^{n} \\ \begin{bmatrix} \lambda_\ell F_{\ell 1} + F_{\ell 2} & \lambda_1 F_{\ell 2} + 2F_{\ell 3} & \dots & \lambda_\ell F_{\ell n_\ell} \end{bmatrix}_{\ell=1}^{n} \\ \vdots \\ \begin{bmatrix} \lambda_\ell^i F_{\ell 1} + i\lambda_\ell^{i-1} F_{\ell 2} + \dots & \lambda_\ell^i F_{\ell 2} + 2i\lambda_\ell^{i-1} F_{\ell 3} + \dots & \dots & \lambda_\ell^i F_{\ell n_\ell} \end{bmatrix}_{\ell=1}^{n} \end{bmatrix}.$$

6. CONCLUSION

In this paper, we have utilized the I/O maps for the purpose of realizing linear continuous time systems. We have provided procedures based on which the state space matrices of a realization can be obtained explicitly and efficiently. Computation of the realization uses the system controllability and observability Gramians and does not require the solution of Lyapunov equations. The procedures developed handle systems with distinct modes as well as those with repeated modes. We have also addressed the question of transforming a given realization to its Jordan canonical form once the realization has been obtained through the input or output map. To this end, the required transformations have been explicitly derived. The numerical examples that have been worked out by the authors (but not provided here for the lack of space) demonstrate that the developed procedures are easy to apply in practice and yield the desired results with a high computational efficiency.

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