Delta-Operator-Based Digital Redesign of Linear Time-Invariant Systems

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Abstract: This paper proposes a delta-operator-based digital redesign (DR) technique. An asymptotic property of the delta-operator-based DR is analyzed. The performance recovery is proved as a sampling time approaches zero.

Keywords: Digital redesign (DR), delta-operator, performance recovery

1. Introduction

Digital redesign (DR) has been gained tremendous attentions as an alternative design tool of a sampled-data control. [1–14]. The term DR involves converting a well-designed analog control into an equivalent digital one maintaining the property of the original analog control system in the sense of state-matching, by which the benefits of both the analog control and the advanced digital technology can be achieved. Lately, a new DR method involving linear matrix inequality (LMI) of a full state-feedback control for linear time-invariant (LTI) systems was introduced [1]. The main ingredient of this technique is to simultaneously take into account a stability guaranteeing algorithm as well as the state-matching condition in a unified framework.

However, the LMI-based DR might not provide a satisfactory performance under the fast sampling limit, i.e., the sampling period tends to zero. The reason seems that there is intrinsic difference between a differential operator $\frac{df(t)}{dt} = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ and the shift operator in that it is hard to reveal the asymptotic connection between these two operators. Motivated by the aforementioned observations, this paper aims at refining the DR problem in [1] under the delta-operator framework.

The rest of the paper is organized as follows: Section 2briefly reviews LTI systems for both analog and digital cases. In Section 3, a refined LMI-based DR method is proposed in delta-operator framework and the performance recovery is investigated. This paper closes with Section 5.

2. Preliminaries

2.1. Analog Control Systems

Consider an LTI control system

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t) \tag{1}$$

where $x_c(t) \in \mathbb{R}^n$ and $u_c(t) \in \mathbb{R}^m$. The subscript 'c' means the analog control, while the subscript 'd' will denote the digital control in the sequel.

Assumption 1: The pair (A, B) is controllable. Furthermore, B has full rank.

Remark 1: The full rank assumption of B is quite mild. If $\text{rank}\,(B) \neq m$, an input can be generated from the combination of the others, so is redundant. In conclusion, deleting linearly dependent columns of B and the corresponding inputs will not affect the control of the systems [18, Ch. 6.2.1].

We here consider an exponentially stabilizing (ES) analog control $u_c(t)$ as

$$u_c(t) = K_c x_c(t) \tag{2}$$

where K_c is a given analog control gain matrix. The dynamics (1) closed by (2) is then written as

$$\dot{x}_c(t) = (A + BK_c)x(t). \tag{3}$$

The discrete model of (3), is then

$$x_c(kT + T) = \phi x_c(kT) \tag{4}$$

where $\phi = e^{(A+BK_c)T}$ and $T \in \mathbb{R}_{>0}$ is a sampling period.

2.2. Digital Control Systems

Consider a desired digital LTI control system represented by

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t) \tag{5}$$

where $u_d(t) = u_d(kT)$ is a piecewise constant control to be determined during the time interval $[kT, kT + T)$, $k \in \mathbb{Z}_{\geqslant 0}$. The discrete model of (5) is then

$$x_d(kT + T) = Gx_d(kT) + Hu_d(kT) \tag{6}$$

where $G = e^{AT}$ and $H = \int_{kT}^{kT+T} e^{A(kT+T-\tau)}\, Bd\tau$.

Remark 2: It is noted that the controllability of (G, H) can still be preserved with a proper choice of the nonpathological sampling period T [19, Th. 3.2.1].

Lemma 1: If $\text{rank}\,(B) = m$, then so $\text{rank}\,(H) = m$ for any $T \in \mathbb{R}_{>0}$.

In this study, the redesigned digital control takes the relation

$$u_d(t) = K_d x_d(kT) \tag{7}$$

for the time interval $[kT, kT + T)$, $k \in \mathbb{Z}_{\geqslant 0}$. The controlled system (6) closed by (7) is then described by

$$x_d(kT + T) = (G + HK_d)x_d(kT). \tag{8}$$

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3. Delta-Operator-Based DR

Our goal is to find K_d so that the closed-loop state $x_d(t)$ matches the closed-loop state $x_c(t)$ at every sampling instant as closely as possible, with guaranteed stability. To this end, we formulate the following DR problem.

Problem 1: Given the well-designed analog ES control gain matrix K_c for (2), find the digital control gain matrix K_d for (7) such that the followings are satisfied:

1) The state $x_d(kT)$ of the discrete-time representation (8) of the digitally controlled system (5) with (7) matches the state $x_c(kT)$ of the discrete-time representation (4) of the analogously controlled system (3) at $t = kT$, $k \in \mathbb{Z}_{>0}$, as closely as possible.
2) The hybrid system (5) controlled by (7) is stable.

In [1], by comparing (4) and (8) to realize $x_c(kT+T) = x_d(kT+T)$ under the assumption $x_c(kT) = x_d(kT)$, K_d was numerically synthesized for $\|\phi - G - HK_d\|$ to be a minimizer in the induced 2-norm sense. As a result, the redesigned digital control has provided a better performance with a wider range of sampling period than the previous works [8, 9]. However, it is interesting to note that when T approaches zero, a usually expected convergence of the state-matching performance of the digital control system to the given analog one may not be realized. It is easy to see that $\phi \to I$, $G \to I$, and $H \to \begin{bmatrix}0\end{bmatrix}_{n\times m}$ as $T \to 0$, which signifies the eigenvalues of ϕ and $G + HK_d$ gather around one thereby weakens the numerical robustness of the related convex optimization problem. This is the reason why K_d may not converge to K_c by the method.

Our strategy to refine the DR in [1] under the fast sampling limit is to use the delta-operator in the LMI framework. It has been observed that for fast sampling limit, the delta-operator model is much less sensitive to arithmetic roundoff errors than their counterpart's based on the shift operator [15, 16].

Application of the delta-operator leads the pointwise discrete-time representations (4) and (8) to two discrete-time systems:

$$\delta x_c(kT) = \phi_\delta x_c(kT) \tag{9}$$

and

$$\delta x_d(kT) = (G_\delta + H_\delta K_d)x_d(kT) \tag{10}$$

where $\phi_\delta = \frac{\phi - I}{T}$, $G_\delta = \frac{G-I}{T}$, and $H_\delta = \frac{H}{T}$.

The second objective in Problem 1 can be handled in a delta-operated discrete manner by virtue of the following proposition.

Proposition 1: Suppose (10) is ES; then the zero equilibrium points $x_{d_{\text{eq}}} = \begin{bmatrix}0\end{bmatrix}_{n\times 1}$ of the hybrid digital control system (5) with (7) is also ES.

Now, we are in position to present our main result.

Theorem 1: If there exist matrices $Q = Q^T \succ 0$ and M_d with appropriate dimensions, and possibly small positive scalar γ_1 such that the following minimization problem (MP) has solutions

MP: $\underset{Q, M_d}{Minimize}$ trace(γ_1) subject to

$$\begin{bmatrix} -\gamma_1 Q & (\bullet)^T \\ \phi_\delta Q - G_\delta Q - H_\delta M_d & -\gamma_1 I \end{bmatrix} \prec 0 \tag{11}$$

$$\begin{bmatrix} QG_\delta^T + H_\delta^T M_d^T + G_\delta Q + H_\delta M_d & (\bullet)^T \\ G_\delta Q + H_\delta M_d & -\frac{1}{T}Q \end{bmatrix} \prec 0 \tag{12}$$

then, the state $x_d(kT)$ of the delta-operated discrete-time representation (10) of (5) controlled by (7) closely matches the state $x_c(kT)$ of the delta-operated discrete-time representation of the analogously controlled system (9), and (10) is ES in the sense of Lyapunov, where $(\bullet)^T$ denotes the transposed element in symmetric positions.

Proof: First, introducing a free matrix variable X having full column rank n, we have

$$\begin{aligned} \|\phi_\delta - G_\delta - H_\delta K_d\|_2 &\leqslant \widehat{\gamma}_1 \\ &= \gamma_1 \|X\|_2 \end{aligned} \tag{13}$$

where $\gamma_1 = \widehat{\gamma}_1 / \|X\|_2 \in \mathbb{R}_{\geqslant 0}$. Picking X such that $P \succ X^T X$ and by the induced 2-norm definition, (13) holds if

$$(\phi_\delta - G_\delta - H_\delta K_d)^T(\phi_\delta - G_\delta - H_\delta K_d) \prec \gamma_1^2 P.$$

Performing the Schur complement and the congruence transformation with diag$\{P^{-1}, I\}$, and denoting $Q = P^{-1}$, $K_d P^{-1} = M_d$ yields (11). LMI (12) directly follows from the standard Lyapunov ES theorem. This completes the proof of the theorem. ■

4. Performance Recovery of DR

In this section, we see that the redesigned digital control in Theorem 1 recovers the performance of the given analog control as the sampling period tends to zero.

Theorem 2: The digital control gain K_d redesigned by Theorem 1 uniquely converges to K_c, as T approaches zero.

Proof: By using Assumption 1, Lemma 1, and Theorem 1, it can be proved. ■

Theorem 3: Assume that $x_d(0) = x_c(0) = x_0$ and Theorem 1 is fulfilled; then the analogously controlled trajectory is recovered via the redesigned digital control as T approaches zero.

Proof: Let $e(t) = x_c(t) - x_d(t)$. By the ES of $x_{\text{eq}} = \begin{bmatrix}0\end{bmatrix}_{n\times 1}$ of both analog and digital control systems, there exists $\kappa = \kappa(\epsilon) \in \mathbb{R}_{>0}$ such that $\|e(t)\| \leqslant \epsilon$, $t \in [\kappa, \infty)$. The triangle inequality admits $\|e(t)\| \leqslant \|e(kT)\| + \|x_c(t) - x_c(kT)\| + \|x_d(t) - x_d(kT)\|$. By using the Gronwall–Bellman inequality, one can always choose T_1 and T_2 such that $\alpha_1(T_1) \leqslant \frac{\epsilon}{3}$ and $\alpha_2(T_2) \leqslant \frac{\epsilon}{3}$, respectively, for any $\epsilon \in \mathbb{R}_{>0}$. In view of this, it is sufficient to prove that $\|e(kT_3)\| \leqslant \frac{\epsilon}{3}$ for all $k \in \left[0, \lfloor \frac{\kappa}{T_3} \rfloor\right] \subset \mathbb{Z}_{\geqslant 0}$, for some $\kappa \in \mathbb{R}_{>0}$, which is easily shown by consulting Theorem 1 and 2. Now, we can conclude that, given any $\epsilon \in \mathbb{R}_{>0}$, if T is chosen to satisfy $T \leqslant \min\{T_1, T_2, T_3\}$ then $\|e(t)\| \leqslant \epsilon$, for all $t \in \mathbb{R}_{\geqslant 0}$. This completes the proof. ■

5. Closing Remarks

This paper proposed the delta-operator-based DR. A constructive DR condition was provided for a state-feedback control of an LTI system. The performance recovery of the digital control redesigned in this paper to analog control under a fast sampling limit was rigorously proved.

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