Multirate Control of Takagi-Sugeno Fuzzy System

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Abstract: In this paper, a new dual-rate digital control technique for the Takagi-Sugeno (T-S) fuzzy system is suggested. The proposed method takes account of the stabilizablity of the discrete-time T-S fuzzy system at the fast-rate sampling points. Our main idea is to utilize the lifted control input. The proposed approach is to obtain the dual-rate discrete-time T-S fuzzy system by discretizing the overall dynamics of the T-S fuzzy system with the lifted control, and then to derive the sufficient conditions for the stabilization in the sense of the Lyapunov asymptotic stability for this system. An example is provided for showing the feasibility of the proposed discretization method.

Keywords: Digital control, Takagi-Sugeno (T-S) fuzzy system, stability, dual-rate sampling, linear matrix inequalities (LMIs).

1. Introduction

Many industrial control systems consist of an analog plant and a digital controller interconnected via analog to digital (A/D) and digital to analog (D/A) converters. Owing to the recent development of the microprocessor and its interfacing hardware, the digital controller is popularly utilized for controlling complex dynamical systems such as aircrafts [1], robots [4], hard disc drivers [5], and chaotic systems [2, 3, 6–10].

In practice, there are not all systems in which the A/D and the D/A conversions are made uniformly at one single rate. The faster D/A converter is used to take into account of the effects of the intersampling behavior of the system. There are also situations where the converse is true. For example, it is difficult to implement antialiasing filters with long time constants using analog technique. In such cases, it is much easier to apply the faster A/D. Above and beyond these causes, formulating the faster D/A or the faster A/D aries from the hardware restrictions [14]. In both case, A/D and D/A converters are operated at different rates. This is called as *multirate control system.*

There have been fruitful researches in the digital control system focusing on the multirate sampling. Systems with multirate sampling were first analyzed in Kranc [15]. Additional researches are given in Jury [16, 17], and Kalman and Bertram [18]. More recent work on multirate systems is concerned with e.g. system analysis and stability [14, 19–21], optimal control of multirate systems with a quadratic cost function [22–24], and $\mathcal{H}_\infty$ control of multirate systems [25, 26]. It is noted that these multirate digital control schemes basically work only for a class of linear systems. For that reason, it is highly demanded to develop the intelligent multirate digital control for complex nonlinear systems.

Motivated by the above observations, this paper aims at merging the Takagi-Sugeno (T-S) fuzzy model-based digital control and the multirate control technique for a class of nonlinear systems. The main contribution of this paper is to derive some sufficient conditions, in terms of the linear matrix inequalities (LMIs), such that the digitally controlled system is asymptotically stable at every intersampling points. Specifically, our main idea is to utilize the lifted control input. The proposed approach is to obtain the dual-rate discrete-time T-S fuzzy system by discretizing the overall dynamics of the T-S fuzzy system with the lifted control, and then to derive the sufficient conditions for the stabilization in the sense of the Lyapunov asymptotic stability for this system. An example is provided for showing the feasibility of the proposed discretization method.

This paper is organized as follows: In Section 2., the local discretization of the continuous-time T-S fuzzy system is reviewed, and the global discretization problem is formulated. Section 3.discusses a new global discretization of the continuous-time T-S fuzzy system. In Section 4, a chaotic Lorenz system is used to demonstrate the effectiveness of our discretization method. The paper is concluded in Section 5.

2. Preliminaries and Problem Description

The T-S fuzzy model is described by fuzzy IF-THEN rules, which represent local linear input-output relations of a nonlinear system. Consider the ith fuzzy rule of a SD T-S fuzzy model with the sampling time T_s governed by

$$R_i : \text{IF } z_1(t) \text{ is about } \Gamma_{i1} \text{ and } \cdots \text{ and } z_p(t) \text{ is about } \Gamma_{ip}$$
$$\text{THEN } \frac{d}{dt}x(t) = A_i x(t) + B_i u(t) \tag{1}$$

where $R_i, i \in \mathcal{I}_q = \{1, 2, \ldots, q\}$, is the ith fuzzy rule, $z_h(t), h \in \mathcal{I}_p = \{1, 2, \ldots, p\}$, is the hth premise variable, $\Gamma_{ih}, (i, h) \in \mathcal{I}_q \times \mathcal{I}_p$, is the fuzzy set, and $u(t) = u(kT_s)$ is the piecewise-constant control input vector to be determined time interval $[kT_s, kT_s + T_s)$. Given a pair $(x(t), u(t))$, using the center-average defuzzification, product inference, and singleton fuzzifier, the overall dynamics of the SD T-S fuzzy model (1) is described by

$$\frac{d}{dt}x(t) = \sum_{i=1}^{q} \theta_i(z(t)) \left(A_i x(t) + B_i u(t)\right) \tag{2}$$

This work was supported in part by the Korea Science and Engineering Foundation (Project number: R05-2004-000-10498-0)

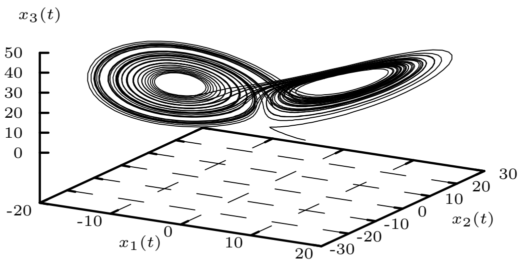

Fig. 1. Trajectory of the Lorenz system.

where $w_i(z(t)) = \prod_{h=1}^{p} \Gamma_{ih}(z_h(t))$, $\theta_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{q} w_i(z(t))}$, and $\Gamma_{ih}(z_h(t))$ is the grade of membership of $z_h(t)$ in Γ_{ih}. Based on the PDC [12, 13], we consider the following fuzzy digital control law for the fuzzy model (2):

$$R_i : \text{IF } z_1(kT_s) \text{ is about } \Gamma_{i1} \text{ and } \cdots \text{ and } z_p(kT_s) \text{ is about } \Gamma_{ip}$$
$$\text{THEN } u(t) = K_{di}x(kT_s) \tag{3}$$

for $t \in [kT_s, (k+1)T_s)$. The overall state feedback fuzzy-model-based digital control law is represented by

$$u(t) = \sum_{i=1}^{q} \theta_i(z(kT_s))K_i x(kT_s) \tag{4}$$

Problem 1: Because the fuzzy model (2) is a hybrid system which involves both continuous-time and discrete-time signals, in general, stabilizing controller at the intersampling points do not exist. The aim of this paper is to design the digital control law (4) such that the closed-loop system is asymptotically stable at every intersampling points.

3. Main Results

To remedy the unfortunate intersample ripple problem, we apply the the fast discretization technique[11], which considers the fast sampling time $T_f = \frac{T}{n}$, to the sampled-data T-S fuzzy system (2). The fast discretization leads to a dual-rate discrete-time system which can be lifted to a single-rate discrete-time system. Specifically, we first connect the fast-sampling operator $\mathcal{S}_{T_f}$ and the fast-hold operator $\mathcal{H}_{T_f}$ with the subinterval $[kT + lT_f, kT + (l+1)T_f)$, $l = 0, 1, \ldots, n-1$, to the sampled-data T-S fuzzy system (2). This leads to a time-varying discrete-time system because of two sampling rates. To remedy this, we invoke the discrete-time lifting.

Assumption 1: [10] Suppose that the firing strength $\theta_i(t)$ for $t \in [kT_s, (k+1)T_s)$ is $\theta_i(kT_s)$. That is

$$\theta_i(t) \approx \theta_i(kT_s) \tag{5}$$

Then, the nonlinear matrices $\sum_{i=1}^{q} \theta_i(z(t))A_i$ and $\sum_{i=1}^{q} \theta_i(z(t))B_i$ of (2) can be approximated as the piecewise constant matrices $\sum_{i=1}^{q} \theta_i(z(kT_s))A_i$ and $\sum_{i=1}^{q} \theta_i(z(kT_s))B_i$, respectively.

Theorem 1: The sampled-data T-S fuzzy system (2) can be converted to the following pointwise dynamical behavior

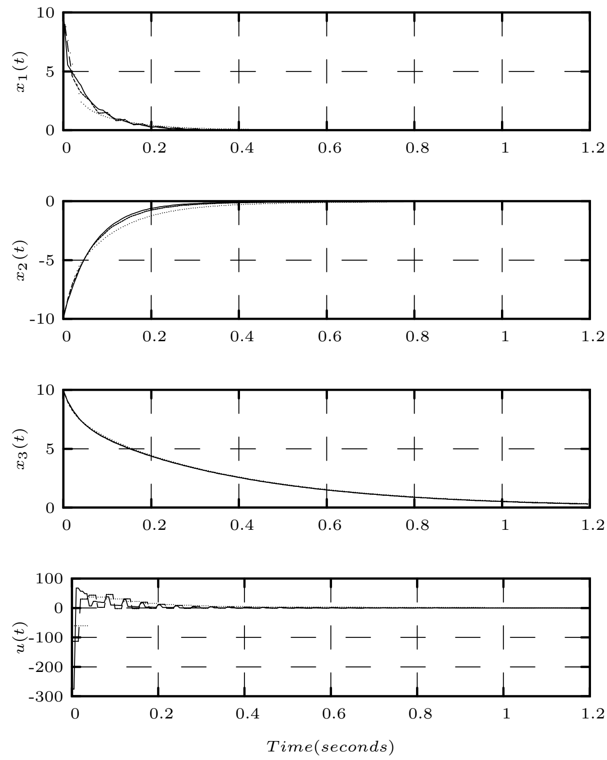

Fig. 2. The time responses of the controlled Lorenz system ($T_s = 0.04$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

with a slow sampled system and a lifted sampled input:

$$x[k+1] = G(\theta[k])x[k] + \tilde{H}(\theta[k])\tilde{u}[k] \tag{6}$$

where $x[k] = x(kT_s)$, $u[k] = u(kT_s)$, $\theta[k] = \theta(z(kT_s))$, $G(\theta[k]) = G_f^n(\theta[k]) = \left(\sum_{i=1}^{q} \theta_i[k]G_{fi}\right)^n$, $H(\theta[k]) = \left(G_f^{n-1}(\theta[k]) + G_f^{n-2}(\theta[k]) + \cdots + I\right) H_f(\theta[k])$, $H_f(\theta[k]) = \sum_{i=1}^{q} \theta_i(z(kT_s))H_{fi}$, $G_{fi} = \exp(A_i T_f)$, $H_{fi} = (G_{fi} - I)A_i^{-1}B_i$, a lifted sampled input $\tilde{u}[k]$ is defined as

$$\tilde{u}[k] = \begin{bmatrix} \tilde{u}_1[k] \\ \tilde{u}_2[k] \\ \vdots \\ \tilde{u}_n[k] \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} u(lT_f) \\ u((l+1)T_f) \\ \vdots \\ u((l+n-1)T_f) \end{bmatrix} \tag{8}$$

and the matrices $G(\theta[k])$ and $\tilde{H}(\theta[k])$ are given by

$$\begin{aligned} &\left[G(\theta[k]), \tilde{H}(\theta[k])\right] \\ &= \Big[\; G_f^n(\theta[k]), G_f^{n-1}(\theta[k])H_f(\theta[k]) \quad G_f^{n-2}(\theta[k])H_f(\theta[k]) \\ &\quad \cdots \quad H_f(\theta[k]) \;\Big] \end{aligned} \tag{9}$$

Corollary 1: The fast-sampled discrete-time system of (2) is obtained as follows:

$$x[l+1] = G_f(\theta[l])x[l] + H_f(\theta[l])u[l] \tag{10}$$

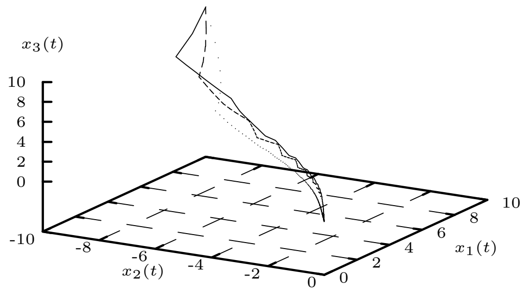

Fig. 3. The trajectories of the controlled Lorenz system ($T_s = 0.04$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

where $x[l] = x(lT_f)$ and $\theta[l] = \theta(lT_f)$.

Proof: When $n = 1$ and $T_s = T_f$, it can be straightforwardly proved by Theorem 1. ■

Now, we derive the stability conditions for the dual-rate T-S fuzzy system (6). Consider the open-loop system for (6).

$$x[k+1] = G(\theta[k])x[k] \tag{11}$$

The following theorem gives a set of conditions for ensuring the stability of (11)

Theorem 2: The equilibrium of (11) is globally asymptotically stable in the sense of Lyapunov stability criterion if there exists a common positive definite matrix P such that

$$G_{fi}^T P G_{fi} - P \prec 0 \quad i \in [1, q] \tag{12}$$

Remark 1: From Theorem 2, we know that if $G_f(\theta[l])$ is globally asymptotically stable, so is $G(\theta[k])$. This is very useful property for the design of digital controller.

Our main objective is to construct the stabilizing controller for (2) at fast-rate sampling points. We first design a stabilizing controller for the fast-sampled discrete-time system, and then convert the controlled system into (6).

We consider the following state feedback fuzzy control law for (10):

$$u[l] = \sum_{i=1}^{q} \theta[l] K_i x[l] \tag{13}$$

Then, the closed-loop system can be rewritten as

$$x[l+1] = \sum_{i=1}^{q}\sum_{j=1}^{q} \theta_i[l]\theta_j[l]\left(G_{fi} + H_{fi}K_j\right)x[l] \tag{14}$$

The following theorem provides the sufficient conditions for the stabilization in the sense of the Lyapunov asymptotic stability for (2).

Theorem 3: For the dual-rate T-S fuzzy system (6), the closed-loop system under the state feedback controller law

$$\tilde{u}[k] = \begin{bmatrix} K(\theta[k]) \\ K(\theta[k])\left(G_f(\theta[k]) + H_f(\theta[k])K(\theta[k])\right) \\ \vdots \\ K(\theta[k])\left(G_f(\theta[k]) + H_f(\theta[k])K(\theta[k])\right)^{n-1} \end{bmatrix} x[k] \tag{15}$$

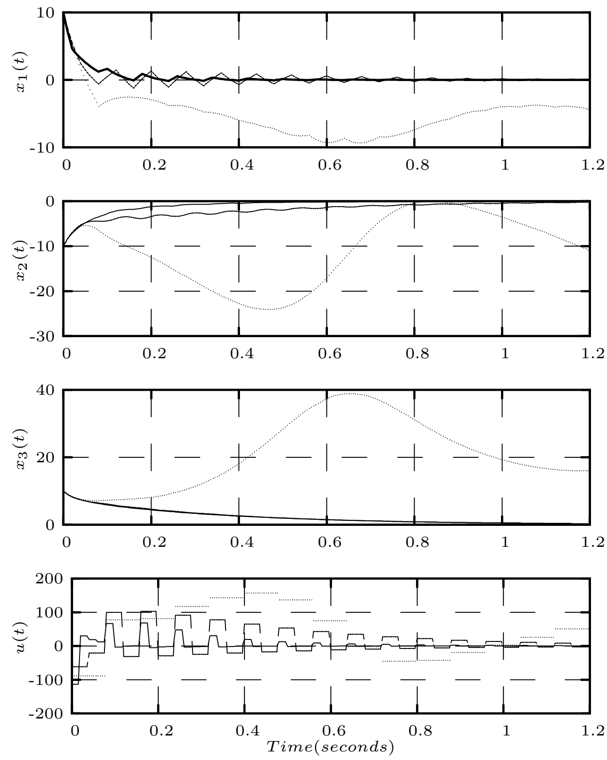

Fig. 4. The time responses of the controlled Lorenz system ($T_s = 0.08$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

is globally asymptotically stabilizable in the sense of Lyapunov stability criterion if there exist symmetric positive definite matrix Q and constant matrix F such that (16), (17) shown at the top of the next page, where $*$ denotes the transposed element in symmetric position.

4. Computer Simulations

In this section, we use the results in Section 3to discretize the continuous-time T-S fuzzy system, which is the fuzzy model of the chaotic Lorenz equation. The Lorenz equation is given by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -\sigma x_1(t) + \sigma x_2(t) \\ r x_1(t) - x_2(t) - x_1(t)x_3(t) \\ x_1(t)x_2(t) - b x_3(t) \end{bmatrix} \tag{18}$$

where $\sigma, r, b > 0$ are parameters (σ is the Prandtl number, r is the Rayleigh number, and b is a scaling constant). The corresponding T-S fuzzy model of the system in (18) is expressed as follows:

$$\begin{aligned} &R_1 : \text{IF } x_1(t) \text{ is about } \Gamma_{11}, \text{ THEN } \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = A_1 \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \\ &R_2 : \text{IF } x_1(t) \text{ is about } \Gamma_{21}, \text{ THEN } \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = A_2 \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \end{aligned} \tag{19}$$

$$\begin{bmatrix} \left(\frac{G_{fi}Q+H_{fi}F_i-Q}{T_f}\right)^T + \frac{G_{fi}Q+H_{fi}F_i-Q}{T_f} & * \\ \frac{G_{fi}Q+H_{fi}F_i-Q}{\sqrt{T_f}} & -Q \end{bmatrix} \prec 0 \quad i \in [1,q] \tag{16}$$

$$\begin{bmatrix} \left(\frac{G_{fi}Q+H_{fi}F_j+G_{fj}Q+H_{fj}F_i-2Q}{2T_f}\right)^T + \frac{G_{fi}Q+H_{fi}F_j+G_{fj}Q+H_{fj}F_i-2Q}{2T_f} & * \\ \frac{G_{fi}Q+H_{fi}F_j+G_{fj}Q+H_{fj}F_i-2Q}{2\sqrt{T_f}} & -Q \end{bmatrix} \prec 0 \quad i < j \in [1,q] \tag{17}$$

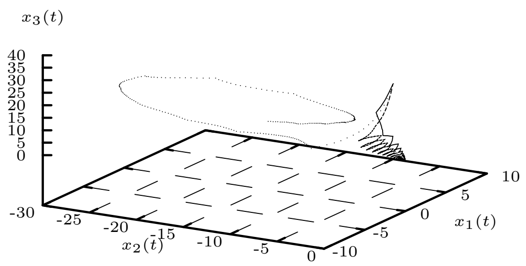

Fig. 5. The trajectories of the controlled Lorenz system ($T_s = 0.08$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

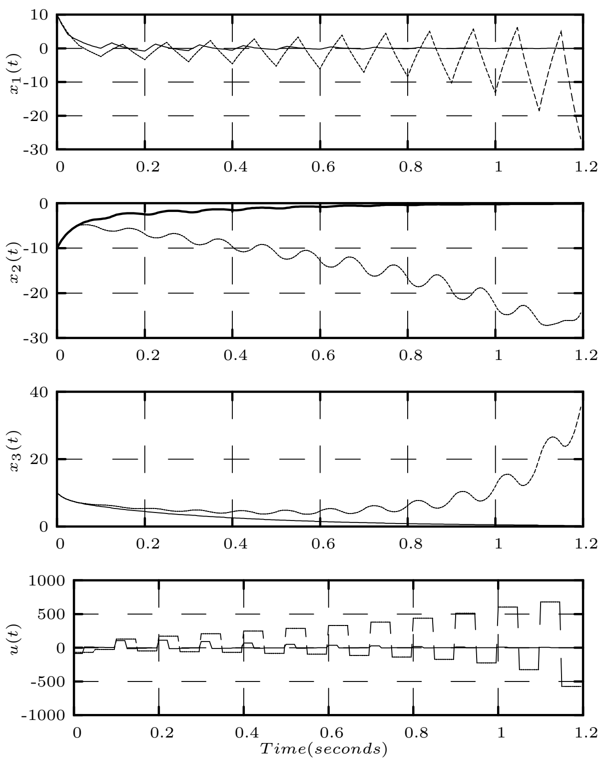

Fig. 6. The time responses of the controlled Lorenz system ($T_s = 0.1$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

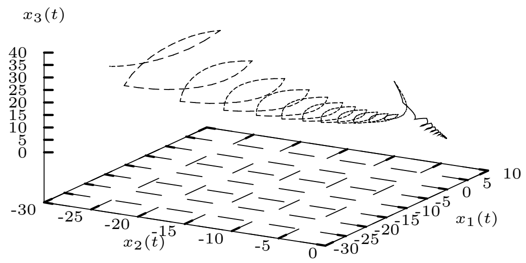

Fig. 7. The trajectories of the controlled Lorenz system ($T_s = 0.1$ sec., dotted line: $n = 1$; dashed line: $n = 2$; solid line: $n = 4$).

where

$$A_1 = \begin{bmatrix} \sigma & -\sigma & 0 \\ r & -1 & -x_{1min} \\ 0 & x_{1min} & -b \end{bmatrix}, \quad A_2 = \begin{bmatrix} \sigma & -\sigma & 0 \\ r & -1 & -x_{1max} \\ 0 & x_{1max} & -b \end{bmatrix}, \tag{20}$$

and the membership functions are

$$\Gamma_1^1(x_1(t)) = \frac{-x_1(t) + x_{1max}}{x_{1max} - x_{1min}}, \quad \Gamma_1^2(x_2(t)) = \frac{x_1(t) - x_{1min}}{x_{1max} - x_{1min}}. \tag{21}$$

where Γ_{ij} are positive semi-definite for all $x \in [x_{1min}, x_{1max}] = [-20, 30]$.

First, we simulate the continuous-time T-S fuzzy system. The input matrices are arbitrary chosen as

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

where preserve the controllability of the system. Figure 1 shows the trajectory of the T-S fuzzy system of the Lorenz system with the input $u(t) = 10\sin(10t)$, the parameter choice $(\sigma, r, b) = (10, 28, 8/3)$, and initial condition $x(0) = [10, -10, 10]^T$.

For the T-S fuzzy system (19) with (22), we seek to a stabilizing dual-rate digital controller (15), where $n = 1$, 2, and 4. Applying Theorem 3 yields the digital gain matrices K_{di}

for the sampling $T_s = 0.04$ sec., as

with $n = 1$,

$$K_1 = \begin{bmatrix} -31.2155 & -21.7680 & -13.3059 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -30.0575 & -16.8091 & 18.1496 \end{bmatrix}$$

with $n = 2$,

$$K_1 = \begin{bmatrix} -55.8677 & -39.9455 & -13.6820 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -55.6458 & -37.7265 & 18.7637 \end{bmatrix}$$

with $n = 4$,

$$K_1 = \begin{bmatrix} -105.1277 & -72.8969 & -14.2594 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -105.0382 & -72.2410 & 17.9034 \end{bmatrix} \quad (23)$$

Figure 2 and 3 report that all trajectories are guided to the equilibrium points at origin.

Another relatively longer sampling $T_s = 0.08$ sec. is chosen so as to show the superiority of the proposed method to the single-rate control ($n = 1$) in the stabilizability. Based on Theorem 3, the digital control gain matrices are obtained as follows:

with $n = 1$,

$$K_1 = \begin{bmatrix} -17.0985 & -8.2747 & -10.9188 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -14.0584 & -1.2786 & 10.9158 \end{bmatrix}$$

with $n = 2$,

$$K_1 = \begin{bmatrix} -31.2155 & -21.7680 & -13.3059 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -30.0575 - 16.809118.1496 \end{bmatrix}$$

with $n = 4$,

$$K_1 = \begin{bmatrix} -55.8677 - 39.9455 - 13.6820 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -55.6458 - 37.726518.7637 \end{bmatrix} \quad (24)$$

Figure 2 and 3 depict the trajectories and the time responses of the digitally controlled system. As shown in these figures, the single-rate digitally controlled system is not stable in spite of obtaining the feasible gain matrices. On the other hand, the dual-rate controllers with $T_f = \frac{T_s}{2}$ and $T_f = \frac{T_s}{4}$ stabilize the given system.

Finally, we examine the case of a large sampling $T_s = 0.1$ sec.. From Theorem 3, it is impossible to obtain the feasible gain matrices for the single-rate digital control ($n = 1$). The dual-rate gain matrices can be given by

with $n = 2$,

$$K_1 = \begin{bmatrix} -26.1471 & -17.1350 & -13.2475 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -24.2624 & -10.8414 & 17.2503 \end{bmatrix}$$

with $n = 4$,

$$K_1 = \begin{bmatrix} -45.9875 & -33.0960 & -13.5411 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -45.5765 & -30.0424 & 18.8604 \end{bmatrix} \quad (25)$$

Figure 6 and 7 shows that the time responses and the trajectories of the dual-rate controlled systems. Compared with $n = 2$, the stbilizability of the given system can be well guaranteed, because the digital control is ensured at the intersample points ($\frac{T_s}{4}$).

It is noted that the proposed method guarantees the stability of the controlled system in much wider range of sampling period than the single-rate digital method in which may fail to stabilize the system especially for relatively longer sampling period, which is major advantage of the proposed method. This is because the proposed dual-rate control is ensured at intersample points, whereas the other approach does not.

5. Closing Remarks

In this paper, a new dual-rate digital control method has been proposed for the T-S fuzzy system, and its validity has been verified through the computer simulations. We have formulated and solved the intersampling stability problem for the fuzzy-model-based sampled-data system. The proposed fast discretization approach leads to the dual-rate T-S fuzzy system which can be lifted to a single-rate discrete-time system. For this system, the stability conditions at the fast-rate sampling points have been derived. Finally, for the digitally controlled T-S fuzzy system, the sufficient stabilization conditions in the sense of the Lyapunov asymptotic stability have been derived. For the given simulations, the results have shown that the proposed discretization method yields the smaller discretization error than the conventional discretiozation method. It indicates the great potential for reliable application of design of the fuzzy-model-based digital controller.

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