Some Properties on Receding Horizon $\mathcal{H}_\infty$ Control for Nonlinear Discrete-time Systems

Choon Ki Ahn, Soo Hee Han, and Wook Hyun Kwon

School of Electrical Engineering & Computer Science, Seoul National University, Seoul , 151-742, Korea
(Tel: +82-2-880-7314; http://cisl.snu.ac.kr; Email:hironaka@cisl.snu.ac.kr)

Abstract: In this paper, we present some properties on receding horizon $\mathcal{H}_\infty$ control for nonlinear discrete-time systems. First, we propose the nonlinear inequality condition on the terminal cost for nonlinear discrete-time systems. Under this condition, noninceasing monotonicity of the saddle point value of the finite horizon dynamic game is shown to be guaranteed. We show that the derived condition on the terminal cost ensures the closed-loop internal stability. The proposed receding horizon $\mathcal{H}_\infty$ control guarantees the infinite horizon $\mathcal{H}_\infty$ norm bound of the closed-loop systems. Also, using this cost monotonicity condition, we can guarantee the asymptotic infinite horizon optimality of the receding horizon value function. With the additional condition, the global result and the input-to-state stable property of the receding horizon value function are also given. Finally, we derive the stability margin for the saddle point value based receding horizon controller. The proposed result has a larger stability region than the existing inverse optimality based results.

Keywords: Receding horizon control, Nonlinear Systems, $\mathcal{H}_\infty$ control, Input-to-state stability, Stability margin

1. Introduction

Receding horizon control (RHC) has been widely investigated as a successful feedback strategy [1], [2], [3], [4], [5], [6]. The basic concept of the receding horizon control is to solve an optimization problem for a finite future horizon at current time and implement the first solution as a current control law. For the closed-loop stability of the RHC, one approach is to impose infinite terminal weighting which is equivalent to setting a zero terminal weighting matrix for the inverse Riccati equation [1], [2]. This is referred to as the terminal equality condition. Since imposing infinite terminal weighting is demanding, use of finite terminal weighting matrices has been investigated [5], [6]. As an alternative approach to finite horizons, an infinite horizon formulation has been explored [3], [4]. However, infinite horizon formulations can also be treated and approximated as finite horizon formulations with appropriate finite terminal weighting matrices as shown in [6] for linear systems and as shown in [7] for nonlinear systems, respectively.

This RHC has been applied to $\mathcal{H}_\infty$ problems in order to combine the practical advantage of the RHC with the robustness of the $\mathcal{H}_\infty$ control. For the closed-loop stability of linear continuous- time systems, the terminal equality condition is proposed in [8], and the terminal inequality condition is in [9] for the monotonicity of the Riccati equation instead that of the saddle point value. A linear discrete-time result is also presented in [10]. In addition, the terminal inequality condition for the non-deceasing monotonicity is presented to include the terminal equality condition in[8] for linear continuous case [11] and for linear discrete case [12], respectively. Receding horizon $\mathcal{H}_\infty$ control schemes are proposed for continuous-time nonlinear systems in [13] and [14]. In these results, the solution of nonlinear receding horizon $\mathcal{H}_\infty$ control is based on the linearization around origin. The set of admissible disturbances is of the special type as $||w_k||^2 \le \frac{1}{\gamma^2}||z_k||^2$. For the nonlinear discrete-time, the result of [15] presents the extended domain of attraction while it has the assumption that the $\mathcal{H}_\infty$ control problem is solvable for the linearized system and the admissible set of disturbance form is taken to be the set of the form as $||w_k||^2 \le \frac{1}{\gamma^2}||z_k||^2$. In [16], a somewhat generalization of [15] is presented.

In this paper, we present some results on the cost monotonicity based receding horizon controller. we propose the cost monotonicity condition on the terminal cost under which the closed-loop internal stability, the infinite horizon $\mathcal{H}_\infty$ norm bound, and the asymptotic infinite horizon optimality of the receding horizon value function are guaranteed. With the additional condition, the global result and the input-to-state stable property of the receding horizon value function are also given. Finally, we derive the stability margin for the cost monotonicity based receding horizon controller. Our result guarantees a larger stability region than the inverse optimality based results developed in continuous-time framework.

The class of nonlinear systems and the basic problem formulation considered in this paper are described in Section 2. In Section 3, some results on the cost monotonicity based receding horizon controller for nonlinear discrete-time systems are presented. In Section 4, we derive the stability margin for the proposed receding horizon controller. The conclusion is given in Section 5.

2. Problem Formulation

We consider the following nonlinear discrete-time systems

$$x_{k+1} = a(x_k) + b(x_k)u_k + g(x_k)w_k = f(x_k, u_k, w_k) \qquad (1)$$

$$z_k = \begin{bmatrix} h(x_k) \\ u_k \end{bmatrix} \qquad (2)$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ and $w_k \in R^p$ are the control input and the external disturbance, respectively. a, b, g, h are sufficiently smooth functions and $a(0) = 0$, $h(0) = 0$. The solution of the differential game will be in sets of piecewise time-varying feedback-type functions. These spaces are the strategy spaces that we shall call as κ and ν.

The finite horizon optimal differential game at time k consists of the minimization with respect to $\kappa(i-k, x_k)$, and the maximization with respect to $\nu(i-k, x_k)$ of the cost function

$$J(x_k, \kappa, \nu, N) = \sum_{i=k}^{k+N-1} (||z_i||^2 - \gamma^2 ||w_i||^2) + V_f(x_{k+N}) \quad (3)$$

where V_f is a smooth nonnegative function with $V_f(0) = 0$ and γ is a positive constant which stands for the disturbance attenuation level.

For any state x_k, if a feedback saddle-point solution exists, we denote the solution as $\kappa^*(i-k, x_k)$ and $\nu^*(i-k, x_k)$ where $k \leq i \leq k + N - 1$. In the following, the optimal value of the finite horizon optimal differential game will be denoted by $V(x_k, N) = J(x_k, \kappa^*, \nu^*, N)$. In receding horizon control, at each time k, the resulting feedback control at state x_k is obtained by solving the finite horizon optimal differential game (FHODG) and setting

$$\kappa^{RH}(x_k) = \kappa^*(0, x_k). \quad (4)$$

According to the standard result of the dynamic programming, we have

$$\begin{aligned} V(x_i) =& \min_u \max_w \Big[h(x_i)^T h(x_i) + u_i^T u_i - \gamma^2 w_i^T w_i + V(x_{i+1}) \Big] \\ =& h(x_i)^T h(x_i) + \kappa^{*T}(i-k, x_k) \kappa^*(i-k, x_k) \\ & - \gamma^2 \nu^{T*}(i-k, x_k) \nu^*(i-k, x_k) + V(a(x_i) \\ & + b(x_i)\kappa^*(i-k, x_k) + g(x_i)\nu^*(i-k, x_k)) \end{aligned} \quad (5)$$

where $k \leq i \leq k + N - 1$ and

$$V(x_{k+N}) = V_f(x_{k+N}) \quad (6)$$

If there exists function $V(x_i)$ satisfying (5) and (6), $\{\kappa^*(i-k, x_k), \nu^*(i-k, x_k)\}$ is the feedback saddle point solution of FHODG [17].

3. Main Results

3.1. Cost Monotonicity and $\mathcal{H}_\infty$ Control with Internal Stability

Now, the monotonicity condition of the saddle point value is investigated.

Theorem 1: If the terminal cost function $V_f(x_k)$ satisfies the following inequality for some $\rho(x_k)$ and $\nu(x_k)$ for all $x_k \neq 0$:

$$\begin{aligned} & h(x_k)^T h(x_k) + \rho^T(x_k)\rho(x_k) - \gamma^2 \nu^T(x_k)\nu(x_k) \\ & + V_f(a(x_k) + b(x_k)\rho(x_k) + g(x_k)\nu(x_k)) - V_f(x_k) \leq 0, \end{aligned} \quad (7)$$

then the saddle point value $V(x_k, N)$ decreases monotonically as follows:

$$V(x_k, N+1) - V(x_k, N) \leq 0, \quad (8)$$

for all positive integer N.

Proof:

Denote the optimal solutions for $V(x_k, N+1)$ and $V(x_k, N)$ by indices 1 and 2, respectively. Subtraction $V(x_k, N)$ from $V(x_k, N+1)$ yields

$$\begin{aligned} & V(x_k, N+1) - V(x_k, N) \\ & = J(x_k, \kappa^*, \nu^*, N+1) - J(x_k, \kappa^*, \nu^*, N) \\ & = \sum_{i=k}^{k+N} (||z_{1i}||^2 - \gamma^2 ||w_{1i}||^2) + V_f(x_{1,k+N+1}) \\ & - \sum_{i=k}^{k+N-1} (||z_{2i}||_i^2 - \gamma^2 ||w_{2i}||^2) - V_f(x_{2,k+N}). \end{aligned}$$

Replacing u_{1i} and w_{2i} with u_{2i} and w_{1i} on $i \in [k,\ k+N-1]$ and using arbitrary $u_{1,k+N} = \rho(x_{k+N})$ and $w_{1,k+N} = \nu(x_{k+N})$, we have

$$\begin{aligned} & V(x_k, N+1) - V(x_k, N) \\ & \leq ||z_{k+N}||^2 - \gamma^2 ||w_{k+N}||^2 + V_f(x_{k+N+1}) - V_f(x_{k+N}) \\ & = h(x_{k+N})^T h(x_{k+N}) + \rho^T(x_{k+N})\rho(x_{k+N}) \\ & - \gamma^2 \nu^T(x_{k+N})\nu(x_{k+N}) + V_f(a(x_{k+N}) + b(x_{k+N})\rho(x_{k+N}) \\ & + g(x_{k+N})\nu(x_{k+N})) - V_f(x_{k+N}) \leq 0 \end{aligned}$$

This completes the proof.

In the following, without the external disturbance ($w_k = 0$), the asymptotical stability result of the proposed receding horizon control law is studied.

Theorem 2: Without the external disturbance, if the monotonicity condition of the saddle point value (7) is satisfied, the system controlled by the proposed receding horizon control law is asymptotically stabilized.

Proof: From the definitions of the finite horizon cost and the saddle point value, we have

$$V(x_k, N) = h^T(x_k)h(x_k) + u_k^T u_k + V(x_{k+1}, N-1)$$

By the result of Theorem 1,

$$V(x_k, N) \geq h^T(x_k)h(x_k) + u_k^T u_k + V(x_{k+1}, N)$$

Since $V(x_k, N)$ is bounded below as

$$V(x_k, N) = \sum_{i=k}^{k+N-1} ||z_i||^2 + V_f(x_{k+N}) \geq 0, \quad (9)$$

$V(x_k, N)$ approaches to the constant number. Thus

$$h^T(x_k)h(x_k) + u_k^T u_k \longrightarrow 0 \text{ as } k \longrightarrow 0, \quad (10)$$

which implies that x_k goes to zero. This completes the proof.

The next two results are used later to show that the proposed RH scheme can guarantee the infinite horizon $\mathcal{H}_\infty$ norm bound.

Lemma 1: The saddle point value of the finite horizon optimal differential game satisfies

$$V(x_k, N) \geq 0 \quad (11)$$

for all nonnegative integer N.

Proof: Given $\nu(i-k, x_k) = 0$, $i \in [k, k+N]$, for every $\kappa^*(i-k, x_k)$, we have $J(x_k, \kappa^*, \nu, N) \geq 0$ and then

$$V(x_k, N) = J(x_k, \kappa^*, \nu^*, N) \geq J(x_k, \kappa^*, \nu, N) \geq 0 \quad (12)$$

This completes the proof.

Lemma 2: Under the monotonicity condition of the saddle point value (7), the saddle point value of FHODG satisfies

$$V(0,N)=0 \tag{13}$$

for all nonnegative integer N.

Proof: If $x_0=0$, because of Theorem 1 and Lemma 1,

$$0\le V(0,N)\le V(0,N-1)\le\cdots\le V(0,0)=V_f(0)=0 \tag{14}$$

This completes the proof.

The following result shows that the proposed receding horizon control scheme can guarantee the infinite horizon $\mathcal{H}_\infty$ norm bound.

Theorem 3: Assume that the terminal cost V_f satisfies the monotonicity condition of the saddle point value (7), the system controlled by the proposed RH scheme guarantees the infinite horizon $\mathcal{H}_\infty$ norm bound with initial state $x_0=0$.

Proof: First, under the assumption of (7), we show that the proposed RH scheme satisfies the dissipative property. Denote the optimal solutions for $V(x_{k+1},N)$ and $V(x_k,N)$ by indices 1 and 2, respectively. Then,

$$\begin{aligned}&V(x_{k+1},N)-V(x_k,N)\\&=\sum_{i=k+1}^{k+N}[h^T(x_{1i})h(x_{1i})+u_{1i}^Tu_{1i}-\gamma^2w_{1i}^Tw_{1i}]+V_f(x_{1,k+N+1})\\&\quad-\sum_{i=k}^{k+N-1}[h^T(x_{2i})h(x_{2i})+u_{2i}^Tu_{2i}-\gamma^2w_{2i}^Tw_{2i}]-V_f(x_{2,k+N})\end{aligned}$$

Replacing u_{1i} and w_{2i} with u_{2i} and w_{1i} on $i\in[k+1,\ k+N-1]$ and using arbitrary $u_{1,k+N}=\rho(x_{k+N})$ and $w_{1,k+N}=\nu(x_{k+N})$, we have

$$\begin{aligned}&V(x_{k+1},N)-V(x_k,N)\\&\le-h^T(x_k)h(x_k)-u_k^Tu_k+\gamma^2w_k^Tw_k+h(x_{k+N})^Th(x_{k+N})\\&\quad+\rho^T(x_{k+N})\rho(x_{k+N})-\gamma^2\nu^T(x_{k+N})\nu(x_{k+N})\\&\quad+V_f(a(x_{k+N})+b(x_{k+N})\rho(x_{k+N})+g(x_{k+N})\nu(x_{k+N}))\\&\quad-V_f(x_{k+N})\\&\le-h^T(x_k)h(x_k)-u_k^Tu_k+\gamma^2w_k^Tw_k\end{aligned} \tag{15}$$

This shows that the system controlled by our RH scheme is dissipative with a storage function $V(x_k,N)$ and the supply rate $-h^T(x_k)h(x_k)-u_k^Tu_k+\gamma^2w_k^Tw_k$. Thus,

$$\begin{aligned}&V(x_\infty,N)-V(x_0,N)\\&\le\sum_{k=0}^{\infty}\Big[-h^T(x_k)h(x_k)-u_k^Tu_k+\gamma^2w_k^Tw_k\Big]\end{aligned}$$

From Lemma 1 and 2, with initial state $x_0=0$, we have

$$0\le\sum_{k=0}^{\infty}\Big[-h^T(x_k)h(x_k)-u_k^Tu_k+\gamma^2w_k^Tw_k\Big]$$

which implies

$$\frac{\sum_{k=0}^{\infty}\left(h^T(x_k)h(x_k)+u_k^Tu_k\right)}{\sum_{k=0}^{\infty}\left(w_k^Tw_k\right)}\le\gamma^2$$

This completes the proof.

3.2. Asymptotic Infinite Horizon Optimality

Consider the nonlinear discrete-time system (1) with the following infinite horizon cost function

$$J_{IH}(x_0,u)=\sum_{i=0}^{\infty}(||z_i||^2-\gamma^2||w_i||^2) \tag{16}$$

In this subsection, we are ready to show that the proposed receding horizon controller can guarantee asymptotic infinite horizon optimality.

Theorem 4: Let $V(x_k,N)$ be the optimal value function for receding horizon dynamic game. And define $V_{IH}(x_k)$ as a solution to the infinite horizon Hamilton-Jacobi-Isaacs equation. Then $V(x_k,N)\to V_{IH}(x_k)$ by the proposed nonlinear receding horizon control scheme.

Proof: Due to the space limitation, we omit the proof.

3.3. Global Result

Under the growth condition of the nonlinear system, it is shown that the proposed controller can guarantee the global result.

Lemma 3: Assume that there exist $K>0$ and $r>0$ such that

$$||x_k||\ge r\Rightarrow||f(x_k,u_k,0)-x_k||\le Kh^T(x_k)h(x_k) \tag{17}$$

Then the saddle point value $V(x_k,N)$ is radially unbounded.

Proof: Let $\overline{x}_k$ be the trajectory of the system (1) corresponding to $(\kappa^*(i,x_0),0)$ at $i\in[0,N]$ and $\overline{x}_0=x_0$. Then we have

$$\begin{aligned}V(x_0,N)&=J(x_0,\kappa^*,\nu^*,N)\\&\ge J(x_0,\kappa^*,0,N)\\&=\sum_{i=0}^{N-1}\Big(h^T(\overline{x}_i)h(\overline{x}_i)+\kappa^{*T}(i,x_0)\kappa^*(i,x_0)\Big)+V_f(\overline{x}_N)\\&\ge\sum_{i=0}^{N-1}h^T(\overline{x}_i)h(\overline{x}_i)\end{aligned} \tag{18}$$

If $||x_0||>r$, then, because the proposed scheme is asymptotically stable by the Theorem 2 in case of $w_k=0$, there exist a constant $\overline{r}>0$ such that

$$||\overline{x}_N||\le\overline{r} \tag{19}$$

(1) Case 1: $r\le||\overline{x}_N||$

$$\begin{aligned}&\sum_{i=0}^{N-1}h^T(\overline{x}_i)h(\overline{x}_i)\\&\ge\sum_{i=0}^{N-1}\frac{1}{K}||f(\overline{x}_i,\kappa^*(i,x_0),0)-\overline{x}_i||^2\\&=\sum_{i=0}^{N-1}\frac{1}{K}||\overline{x}_{i+1}-\overline{x}_i||\ge\frac{1}{K}\Bigg|\Bigg|\sum_{i=0}^{N-1}(\overline{x}_{i+1}-\overline{x}_i)\Bigg|\Bigg|\\&=\frac{1}{K}||\overline{x}_N-\overline{x}_0||\end{aligned} \tag{20}$$

From the definition of $\overline{x}_k$, we have $\overline{x}_0=x_0$. Thus,

$$V(x_0,N)\ge\frac{1}{K}||\overline{x}_N-x_0|| \quad (2$$

(2) Case 2: $r > ||\overline{x}_N||$

In this case, there exists the largest integer M satisfying $||\overline{x}_M|| \geq r$ and $0 < M < N$.

$$
\begin{aligned}
&\sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i) \\
&= \sum_{i=0}^{M} h^T(\overline{x}_i)h(\overline{x}_i) + \sum_{j=M+1}^{N-1} h^T(\overline{x}_j)h(\overline{x}_j) \\
&\geq \sum_{i=0}^{M} h^T(\overline{x}_i)h(\overline{x}_i) \\
&\geq \sum_{i=0}^{M} \frac{1}{K}||f(\overline{x}_i, \kappa^*(i, x_0), 0) - \overline{x}_i||^2 \\
&= \sum_{i=0}^{M} \frac{1}{K}||\overline{x}_{i+1} - \overline{x}_i|| \geq \frac{1}{K}\left|\left| \sum_{i=0}^{M}(\overline{x}_{i+1} - \overline{x}_i)\right|\right| \\
&= \frac{1}{K}||\overline{x}_M - \overline{x}_0|| = \frac{1}{K}||\overline{x}_M - x_0|| \qquad (22)
\end{aligned}
$$

Thus,

$$V(x_0, N) \geq \frac{1}{K}||\overline{x}_M - x_0|| \qquad (23)$$

From case 1 and 2, if $x_0 \to \infty$, $V(x_0, N) \to \infty$. This completes the proof.

The next lemma shows that the condition $h(x_k) \in \mathcal{K}_\infty$ can guarantee the radially unboundedness of the receding horizon value function.

Lemma 4: Assume that $h(x_k) \in \mathcal{K}_\infty$. Then the saddle point value $V(x_k, N)$ is radially unbounded.

Proof: Let $\overline{x}_k$ be the trajectory of the system (1) corresponding to $(\kappa^*(i, x_0), 0)$ at $i \in [0, N]$ and $\overline{x}_0 = x_0$. Then we have

$$
\begin{aligned}
V(x_0, N) &= J(x_0, \kappa^*, \nu^*, N) \\
&\geq J(x_0, \kappa^*, 0, N) \\
&= \sum_{i=0}^{N-1} \Big(h^T(\overline{x}_i)h(\overline{x}_i) + \kappa^{*T}(i, x_0)\kappa^*(i, x_0) \Big) + V_f(\overline{x}_N) \\
&\geq \sum_{i=0}^{N-1} h^T(\overline{x}_i)h(\overline{x}_i) \\
&\geq h^T(\overline{x}_0)h(\overline{x}_0) = h^T(x_0)h(x_0) \qquad (24)
\end{aligned}
$$

If $x_0 \to \infty$ and $h(x_k) \in \mathcal{K}_\infty$, $V(x_0, N) \to \infty$.
This completes the proof.

Theorem 5: Under the monotonicity condition of the saddle point value (7), if the system satisfies the growth condition of the nonlinear system (17) or $h(x_k) \in \mathcal{K}_\infty$, the system controlled by the proposed RH scheme is globally internally stable and has the global infinite horizon $\mathcal{H}_\infty$ norm bound with initial state $x_0 = 0$.

Proof: Under the condition (17), $V(x_k, N)$ can be regarded as the global Lyapunov function. Thus, the global asymptotical stability result with $w_k = 0$ follows from Theorem 2 and Lemma 3. Also, the global $\mathcal{H}_\infty$ norm bound is guaranteed form Theorem 3 and Lemma 3.

3.4. Input-to-state stable (ISS) Properties

Theorem 6: Under the assumption of (7) and (17), if $h(x_k)$ is radially unbounded,the nonlinear systems (1) controlled by the proposed nonlinear receding horizon control law (4) is input-to-state stable (ISS) with respect to w_k.

Proof: From the proof of Theorem 3, we have

$$
\begin{aligned}
V(x_{k+1}, N) - V(x_k, N) &\leq -h^T(x_k)h(x_k) - u_k^T u_k + \gamma^2 w_k^T w_k \\
&\leq -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k \quad (25)
\end{aligned}
$$

If the nonlinear system satisfies the growth condition (17), from Lemma 3, $V(x_k, N)$ is radially unbounded. Since $V(x_k, N)$ is positive definite from Lemma 1, $V(x_k, N)$ is a class $\mathcal{K}_\infty$ function. Thus, for $V(x_k, N) \in \mathcal{K}_\infty$, we can always find class $\mathcal{K}_\infty$ functions α_1 and α_2 satisfying $\alpha_1(|x_k|) \leq V(x_k, N) \leq \alpha_2(|x_k|)$. Since $h^T(x_k)h(x_k) \in \mathcal{K}_\infty$ and $\gamma^2 w_k^T w_k \in \mathcal{K}$, from the definition 3.2 of [18], $V(x_k, N)$ is an ISS-Lyapunov function. Thus, by Lemma 3.5 of [18], the closed-loop system is ISS.

Corollary 1: Assume that the system satisfies the conditions of Theorem 5, it admits a $\mathcal{K}$ asymptotic gain with respect to w_k.

Proof: Using the result of [18], we can obtain Corollay 1.

4. Stability Margin of the Proposed Nonlinear RH Control Law

In this section, we derive the guaranteed stability margin for the proposed nonlinear receding horizon control law. The sufficient condition that guarantees the disk margin is given in terms of the state and the control. Consider the nonlinear system NS of the form given by (1) with

$$y_k = -\phi(x_k) \qquad (26)$$

where $\phi(x_k)$ is such that (1) and (2) has the $\mathcal{H}_\infty$ performance with $u_k = -y_k$. Thus, $\phi(x_k)$ can be regarded as the proposed RHC law (4). Consider the negative feedback interconnection shown in Figure 1 with the uncertainty $\Delta(\cdot)$. In this case, $u_k = -\Delta(y_k)$. Furthermore, we assume that in the nominal case $\Delta(\cdot) = I$ so that the nominal closed-loop system has an L_2-gain less than or equal to γ.

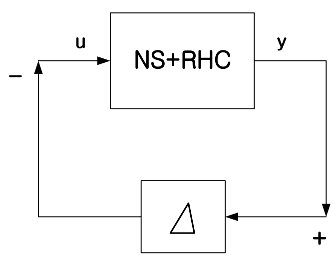

Fig. 1. Feedback Interconnection of $NS + RHC$ and Δ

4.1. $\mathcal{H}_\infty$ Stability Margin

Definition 1: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a $\mathcal{H}_\infty$ gain margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \Delta y$ has an L_2-gain less than or equal to γ for all $\Delta = diag\{k_1, k_2, ..., k_m\}$ where $k_i \in (\alpha, \beta)$, $i = 1, ..., m$.

Definition 2: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a $\mathcal{H}_\infty$ sector margin (α, β) if the negative feedback interconnection of $NS{+}RHC$ and $\Delta(y) = \sigma(y)$ has an L_2-gain less than or equal to γ for all static nonlinearities $\sigma(\cdot)$ such that $\sigma(0) = 0$, $\sigma(y) = [\sigma_1(y_1), ..., \sigma_m(y_m)]^T$, and $\alpha y_i^2 < \sigma(y_i)y_i < \beta y_i^2$ for all $y_i \neq 0$, $i = 1, ..., m$.

Definition 3: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a $\mathcal{H}_\infty$ disk margin $D(\alpha)$ if the negative feedback interconnection of $NS{+}RHC$ and $\Delta(y) = \sigma(y)$ has an L_2-gain less than or equal to γ for all dynamic operator $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state detectable and dissipative with respect to the supply rate as

$$r(u, y) = u^T y - \rho u^T u \tag{27}$$

with a radially unbounded storage function where $\alpha < \rho \in R$.

Remark 1: It is noted that if $NS + RHC$ has a $\mathcal{H}_\infty$ disk margin $D(\alpha)$, then $NS + RHC$ has $\mathcal{H}_\infty$ gain and sector margins (α, ∞).

Now, we are ready to state the main theorem of this section.

Theorem 7: Consider the closed-loop system consisting of the nonlinear systems $NS + RHC$ given by (1), (26). Assume NS and Δ are zero-state observable. Under the monotonicity condition of the saddle point value (7), the proposed nonlinear RH control law has a $\mathcal{H}_\infty$ disk margin $D(\frac{1}{4})$ with input w_k and output $h(x_k)$.

Proof:

Assume that Δ is given by

$$\widetilde{x}_{k+1} = \widetilde{a}(\widetilde{x}_k) + \widetilde{b}(\widetilde{x}_k)\widetilde{u}_k \tag{28}$$

$$\widetilde{y}_k = \widetilde{\phi}(\widetilde{x}_k) \tag{29}$$

with input $\widetilde{u}_k$, output $\widetilde{y}_k$ and is dissipative with respect to the following supply rate

$$r(\widetilde{u}_k, \widetilde{y}_k) = \widetilde{u}_k^T \widetilde{y}_k - \rho \widetilde{u}_k^T \widetilde{u}_k \tag{30}$$

where $\rho \in R$. Also, assume that Δ has the radially unbounded storage function $\widetilde{V}(\widetilde{x}_k)$ with $\widetilde{V}(0) = 0$. Then, we can construct the following dissipative inequality:

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) \leq \widetilde{u}_k^T \widetilde{y}_k - \rho \widetilde{u}_k^T \widetilde{u}_k \tag{31}$$

By the feedback interconnection seen in Figure 1, we have

$$\widetilde{u}_k = y_k$$
$$u_k = -\widetilde{y}_k$$

Thus, (31) becomes

$$\begin{aligned}\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) &\leq -u_k^T y_k - \rho y_k^T y_k && (32)\\ &\leq u_k^T u_k + \frac{1}{4} y_k^T y_k - \rho y_k^T y_k && (33)\\ &= u_k^T u_k - (\rho - \frac{1}{4}) y_k^T y_k && (34)\end{aligned}$$

Using (26), we have

$$\widetilde{V}(\widetilde{x}_{k+1}) - \widetilde{V}(\widetilde{x}_k) \leq u_k^T u_k - (\rho - \frac{1}{4})\phi^T(x_k)\phi(x_k) \tag{35}$$

Under the monotonicity condition of the saddle point value (7), the dissipative inequality (15) is satisfied. Let the new storage function be $V(x_k, \widetilde{x}_k) = V(x_k, N) + \widetilde{V}(\widetilde{x}_k)$. Adding (15) and (35) yields

$$\begin{aligned}&V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k)\\ &\leq -h^T(x_k)h(x_k) - (\rho - \frac{1}{4})\phi^T(x_k)\phi(x_k) + \gamma^2 w_k^T w_k \quad (36)\end{aligned}$$

If we select $\rho \geq \frac{1}{4}$,

$$V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k) \leq -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k \tag{37}$$

Thus,

$$V(x_\infty, \widetilde{x}_\infty) - V(x_0, \widetilde{x}_0) \leq \sum_{k=0}^{\infty} \Big[-h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k\Big] \tag{38}$$

With initial conditions as $x_0 = 0$ and $\widetilde{x}_0 = 0$, we have

$$0 \leq \sum_{k=0}^{\infty} \Big[-h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k\Big] \tag{39}$$

which implies that the overall feedback interconnected system has the infinite horizon $\mathcal{H}_\infty$ performance with input w_k and output $h(x_k)$ as

$$\frac{\sum_{k=0}^{\infty} h^T(x_k)h(x_k)}{\sum_{k=0}^{\infty} w_k^T w_k} \leq \gamma^2$$

From Definition 4, we can conclude that the proposed nonlinear RH scheme has a $\mathcal{H}_\infty$ disk margin $D(\frac{1}{4})$.

Remark 2: The obtained disk margin $D(\frac{1}{4})$ guarantees the larger stability region than the existing result [14] developed in continuous-time framework.

4.2. ISS Stability margin

Definition 4: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a ISS gain margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \Delta y$ is ISS for all $\Delta = diag\{k_1, k_2, ..., k_m\}$ where $k_i \in (\alpha, \beta)$, $i = 1, ..., m$.

Definition 5: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a ISS sector margin (α, β) if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \sigma(y)$ is ISS for all static nonlinearities $\sigma(\cdot)$ such that $\sigma(0) = 0$, $\sigma(y) = [\sigma_1(y_1), ..., \sigma_m(y_m)]^T$, and $\alpha y_i^2 < \sigma(y_i)y_i < \beta y_i^2$ for all $y_i \neq 0$, $i = 1, ..., m$.

Definition 6: Let α, $\beta \in R$ be such that $0 \leq \alpha < 1 < \beta < \infty$. Then the nonlinear system $NS + RHC$ given by (1), (26) is said to have a ISS disk margin $D(\alpha)$ if the negative feedback interconnection of $NS + RHC$ and $\Delta(y) = \sigma(y)$ is ISS for all dynamic operator $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state detectable and dissipative with respect to the supply rate as

$$r(u, y) = u^T y - \rho u^T u \tag{40}$$

with a radially unbounded storage function where $\alpha < \rho \in R$.

Remark 3: It is noted that if $NS + RHC$ has a ISS disk margin $D(\alpha)$, then $NS + RHC$ has ISS gain and sector margins (α, ∞).

At the next theorem, with additional conditions, we show that the saddle point value based controller has a ISS disk margin $D(\frac{1}{4})$.

Theorem 8: At the systems considered in Theorem 6, assume NS and Δ are zero-state observable. Under (7) and (17), if $h(x_k)$ is radially unbounded, the proposed nonlinear RH control law has a ISS disk margin $D(\frac{1}{4})$ with respect to w_k.

Proof: Start with assumptions used in the proof of Theorem 6. Then

$$V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k) \leq -h^T(x_k)h(x_k) - (\rho - \frac{1}{4})\phi^T(x_k)\phi(x_k) + \gamma^2 w_k^T w_k \quad (41)$$

If we select $\rho \geq \frac{1}{4}$,

$$V(x_{k+1}, \widetilde{x}_{k+1}) - V(x_k, \widetilde{x}_k) \leq -h^T(x_k)h(x_k) + \gamma^2 w_k^T w_k \quad (42)$$

From Lemma 3, if we assume (17), $V(x_k, \widetilde{x}_k)$ is radially unbounded. Since $V(x_k, \widetilde{x}_k)$ is positive definite, $V(x_k, \widetilde{x}_k) \in \mathcal{K}_\infty$. Thus, we can find always class $\mathcal{K}_\infty$ functions α_1 and α_2 satisfying $\alpha_1(|x_k|, |\widetilde{x}_k|) \leq V(x_k, \widetilde{x}_k) \leq \alpha_2(|x_k|, |\widetilde{x}_k|)$. Since $h^T(x_k)h(x_k) \in \mathcal{K}_\infty$ with the assumption that $h(x_k)$ is radially unbounded and $\gamma^2 w_k^T w_k \in \mathcal{K}$, from the definition 3.2 of [18], $V(x_k, \widetilde{x}_k)$ is an ISS-Lyapunov function. Thus, by Lemma 3.5 of [18], the closed-loop system is ISS. From Definition 7, we can conclude that the system controlled by the proposed scheme has a ISS disk margin $D(\frac{1}{4})$.

5. Conclusion

In this paper, we propose the cost monotonicity condition on the terminal cost which guarantees noninceasing monotonicity of the saddle point value of the finite horizon dynamic game. With this condition, we can show the closed-loop internal stability, the infinite horizon $\mathcal{H}_\infty$ norm bound, and the asymptotic infinite horizon optimality of the receding horizon value function. With the introduction of the additional condition, the global result and the input-to-state stable property of the receding horizon value function are also given. Finally, we derive the stability margin for the saddle point value based receding horizon controller which guarantees a larger stability region than the inverse optimality based results developed in continuous-time framework.

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