# Control of Age-Dependent Population Systems

A. Mimura* and S. Kubo** and Prof N. Kunimatsu***

*Graduate Schoool of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama, Japan
(Tel: 045-566-1537; Fax: 045-566-1529; Email:mimura-a@nifty.com)
**Graduate Schoool of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama, Japan
(Tel: 045-566-1537; Fax: 045-566-1529; Email:yiu66180@nifty.com)
***Graduate Schoool of Science and Technology, Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama, Japan
(Tel: 045-566-1537; Fax: 045-566-1529; Email:kunimatu@elec.keio.ac.jp)

**Abstract:** In this paper, we at first describe the linear age-dependent population system. In addition, we introduce the nonlinear population system. Using these age-dependent population systems, we evaluate the stability of these age-dependent population systems and determine the optimal birth rates that realize a target distribution which relaxes an aging population. In this paper, we focus on Japan's population and we use the amount of demographic statistics of Japan in year 2000.

**Keywords:** age-dependent population system, stability radius, vector locus of loop transfer function, optimal birth rate, performance index

## 1. Introduction

At the present day, many countries are concerned about population problems. The world's total population is growing rapidly and many advanced nations are concerned about their aging society. Under these circumstances, there are strong needs for quantitative analysis from various aspects toward both each country and the whole world. In particular, Japan nowadays is entering a member of aging countries with falling birth rates and there is a strong need for a major turnaround of age distributions. The population control problem was first investigated thoroughly by [1]. In this paper, we at first describe the linear age-dependent population system. In addition, we introduce the nonlinear population system. Using these age-dependent population systems, our objectives in this paper are to evaluate the stability of these age-dependent population systems and determine the optimal birth rates that realize a target distribution which relaxes an aging population. We focus on Japan's population and we use the amount of demographic statistics of Japan in year 2000.

## 2. Description of the age-dependent population system

### 2.1. The linear population system

2.1.1 The continuous population system

We consider the linear age-dependent population system ([1] 38-44) described by the equation

$$\frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} = -\mu(a)p(a,t), \quad (1)$$

$$p(a,0) = p_0(a), \quad (2)$$

$$p(0,t) = \psi(t) = \beta(t)\int_{a_1}^{a_2} k(a)h(a)p(a,t)da, \quad (3)$$

where $a$ and $t$ denote age and time, $p(a,t)$ denotes the age-dependent population density function, $p_0(a)$ the initial density function, $\mu(a)$ the moratlity rate, and $\psi(t)$ the total number of babies born at time $t$ in unit time(called the birth rate function). $\beta(t)$ is the average birth rate describing the average number of childbirths per female in lifetime, $k(a)$ is the female proportion function and $h(a)$ is the female fertility pattern. $a_1$ and $a_2$ are the minimum and maximum of fertility age of woman. Equation (1) is the continuous modelu of the population evolution process written as a form of partial differential equation. Equation (2) and (3) indicate initial condition and boundary condition respectively.

2.1.2 The discrete population system

At first we note $\mathbf{x}(t)$ as the population state vector, and each component $x_i(t)$ is referred to as the cohort age group of the population.

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix}. \quad (4)$$

The discrete population system ([1] 45-49) can be written in the vector form

$$\mathbf{x}(t+1) = \mathbf{H}(t)\mathbf{x}(t) + \beta(t)\mathbf{B}(t)\mathbf{x}(t), \quad (5)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad (6)$$

where the matrix $\mathbf{H}(t)$ is referred to as the population state matrix of year from $t$ to $t+1$, $\mathbf{B}(t)$ as the birth matrix. Equation (6) is the initial condition. $\mathbf{H}(t)$ and $\mathbf{B}(t)$ can be written as below.

$$\mathbf{H}(t)=\begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1-\mu_1(t) & 0 & & & \\ 0 & 1-\mu_2(t) & & 0 & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & \cdots & 1-\mu_{m-1}(t) & 0 \end{bmatrix} \quad (7)$$

$$\mathbf{B}(t) = \begin{bmatrix} 0 & \cdots & b_{a_1}(t) & \cdots & b_{a_2}(t) & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & \cdots & \cdots & \cdots & & 0 \end{bmatrix}. \quad (8)$$

Here, $b_i(t)$ satisfies $b_i(t) = (1-\mu(0))k(i)h(i)$.

### 2.2. The Nonlinear population system

2.2.1 The continuous population system

We introduce the nonlinear age-dependent population system whose mortality rates $\mu(a)$ depend on population density function $p(a,t)$. The fact that the mortality depends on population density is widely known in biology, so we assume mortality rates to be $\mu(a) + \delta \cdot p(a,t)$. Then the nonlinear population system can be written as

$$\frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} = -\{\mu(a) + \delta \cdot p(a,t)\}p(a,t), \quad (9)$$

$$p(a,0) = p_0(a), \quad (10)$$

$$p(0,t) = \psi(t) = \beta(t)\int_{a_1}^{a_2} k(a)h(a)p(a,t)da, \quad (11)$$

where $\delta(const.)$ denotes a constant parameter which describes the amount of influence that the nonlinear term attributes to the system. The definition of other parameters in this system are identical to those of the linear population system shown in section 2.1.1.

2.2.2 The discrete population system

As well as the linear system, we have to discretize the continuous population system which is described as equation $(9)-(11)$. To compare the linear system with the nonlinear system, we adopt the same method of discretizing as we did in section 2.1.2. So the discrete population system can be described as

$$\mathbf{x}(t+1) = \mathbf{H}(t)\mathbf{x}(t) + \beta(t)\mathbf{B}(t)\mathbf{x}(t), \quad (12)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad (13)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix}, \quad (14)$$

$\mathbf{H}(t) =$

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1-\mu_1(t)+\delta\cdot x_1(t) & & & \\ 0 & \ddots & 0 & \vdots \\ \vdots & & & \\ 0 & \cdots & 1-\mu_{m-1}(t)+\delta\cdot x_{m-1}(t) & 0 \end{bmatrix}, \quad (15)$$

$$\mathbf{B}(t) = \begin{bmatrix} 0 & \cdots & b_{a_1}(t) & \cdots & b_{a_2}(t) & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & \cdots & \cdots & \cdots & & 0 \end{bmatrix}. \quad (16)$$

Here, $b_i(t)$ satisfies $b_i(t) = (1 - \mu(0) + \delta \cdot x_0(t))k(i)h(i)$.

## 3. Stability of the age-dependent population system

### 3.1. Vector locus of loop transfer function

The objective in this section is to evaluate the stability of the age-dependent population system (equation (1)-(3)), applying the method which is shown in [2]. We attempt to figure the vector locus of loop transfer function of the age-dependent population system. Additionally, we calculate the stability radius $r(A:D;E)$ which denotes the stability bound of population system. We make the system description as follows. We introduce the state space $X$ which is a complex ordered Hilbert space $L^2_\mu(0,\infty)$ ($\neq \{0\}$) with the inner product

$$\langle p,q\rangle_\mu := \int_0^\infty e^{2\int_0^a \mu(\xi)d\xi} p(a)\overline{q(a)}da, \quad p,q\in X. \quad (17)$$

The output space $Y$ and the control space $U$ are also chosen to be appropriate complex ordered Hilbert space ($\neq \{0\}$). Let us define the linear operator $A : D(A) \subset X \to X$ by

$$Av := -\partial v - \mu(a)v, \quad (18)$$

$$D(A) := \{v \in L^2_\mu(0,\infty) | Av \in L^2_\mu(0,\infty), v(0) = 0\}. \quad (19)$$

$\partial v$ is the generalized derivative and also $v$ is absolutely continuous.

< **Proposition1** > The population system (1)-(3) can be formulated as the followings. Let the control space be $U = \mathbf{C}$, the output space $Y = \mathbf{C}$ and the state space $X = L^2_\mu(0,\infty)$. Then, the system equation can be written by([2])

$$\frac{d}{dt}p(t) = Ap(t) + Du(t), \quad t>0, \quad p(0) = p_0, \quad (20)$$

$$y(t) = E_{|D(A)}p(t) = \langle c(\cdot), p(t)\rangle_0, \quad (21)$$

$$u(t) = \Delta E_{|D(A)}p(t), \quad (22)$$

where

$$E_{|D(A)} \in L(X,Y) : E_{|D(A)}p = \langle c(\cdot), p(t)\rangle_0,$$
$$c(a) = k(a)h(a)\mu(a) - \partial\{k(a)h(a)\},$$
$$D \in L(U,X) : Dr = rb(\cdot), r \in \mathbf{C},$$
$$b(a) = e^{-\int_0^a \mu(\xi)d\xi},$$
$$\Delta \in L(Y,U) : \Delta r = \beta r, r\in \mathbf{C}, \beta \in \mathbf{R}.$$

Equation (20), (21), (22) denote state equation, output equation, control law respectively. The system transfer function $G(\lambda)$ is defined by

$$G(\lambda)(\cdot) := E_{|D(A)}R(\lambda,A)D(\cdot) = \langle c, (\lambda I - A)^{-1}D(\cdot)\rangle_0, \quad (23)$$

and its norm is

$$||G(\lambda)||_{L(\mathbf{C})} = |\langle c(\cdot), (\lambda I - A)^{-1}b(\cdot)\rangle_0|. \quad (24)$$

This is summarized in Fig.1. So, the system transfer function $G(\lambda)$ can be written as below.

$$G(\lambda) = \langle c(\cdot), (\lambda I - A)^{-1}D(\cdot)\rangle_0 \quad (25)$$

$$= \int_0^{a_m} c(a)e^{-\lambda a - \int_0^a \mu(\xi)d\xi}\frac{1}{\lambda}\left[e^{\lambda a} - 1\right]da \quad (26)$$

$$c(a) = k(a)h(a)\mu(a) - \{k(a)h(a)\}'. \quad (27)$$

As we take $\lambda = \sigma + j\omega$ ($\sigma = 0$), $G(\lambda)$ becomes

$$G(j\omega) = \int_0^{a_m} c(a)\frac{1}{j\omega}\left(e^{j\omega a} - 1\right)e^{-j\omega a - \int_0^a \mu(\xi)d\xi}da$$
$$= \int_0^{a_m}\left\{c(a)\frac{1}{\omega}\sin\omega a + jc(a)\left(\frac{-1}{\omega}\right)(1-\cos\omega a)\right\}$$
$$\times \; e^{-\int_0^a \mu(\xi)d\xi}da, \quad (28)$$

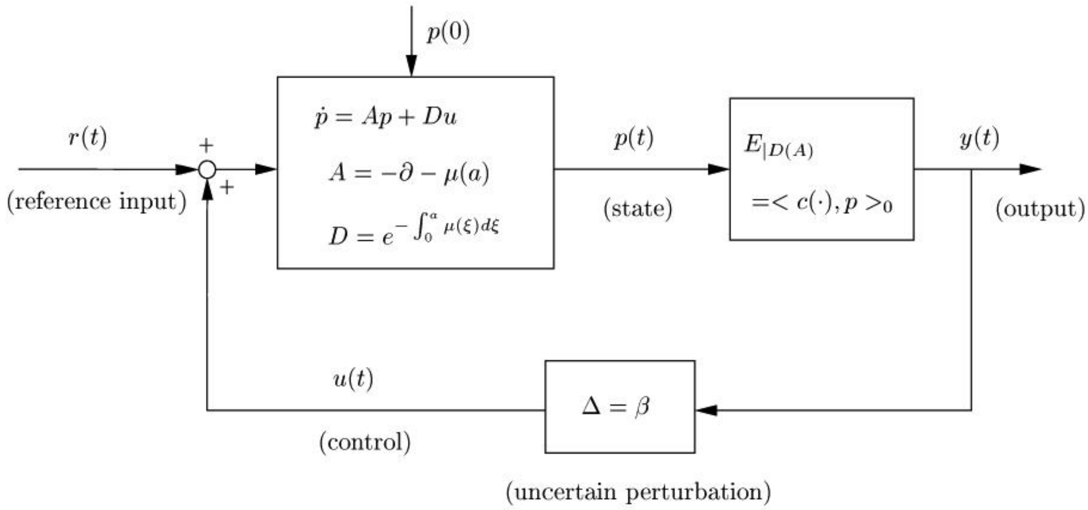

Fig. 1. Block diagram of the system

and $ReG(j\omega)$, $ImG(j\omega)$ become as below.

$$ReG(j\omega)=\int_0^{a_m} c(a)\frac{1}{\omega}\sin\omega a e^{-\int_0^a \mu(\xi)d\xi}da$$
$$ImG(j\omega)=\int_0^{a_m} c(a)\left(\frac{-1}{\omega}\right)(1-\cos\omega a)e^{-\int_0^a \mu(\xi)d\xi}da. \tag{29}$$

Plotting this $G(j\omega)$ from $\omega = 0$ to $\omega = \infty$, we can figure the vector locus of loop transfer function. We used the demographic data(2000) edited by National Institute of Population and Social Security Research(Japan). This is shown in Fig.2.

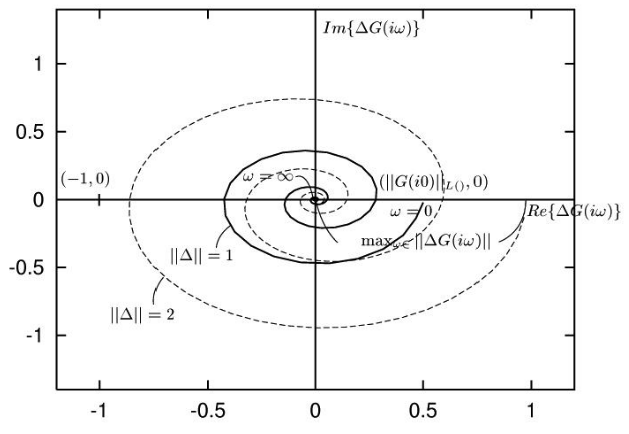

Fig. 2. Vector locus of loop transfer function $\Delta G(\lambda)$ for $\Delta$ : $0 \le ||\Delta|| < r(A : D, E)$, depending on the demographic data $h(a)$,$k(a)$,$\mu(a)$ of Japan-2000.

< **Proposition2** >
Let $A$ be Hurwitz stable. If $\lim_{|\lambda|\to\infty, Re\lambda\ge 0} ||G(\lambda)||_{L(\mathbf{C})}$ exists in $[0,\infty)$, then

$$r(A : D, E) = \{\sup_{\omega\in\mathbf{R}} ||G(i\omega)||\}^{-1} \tag{30}$$

holds([2]). The population system (20)-(22) is Hurwitz stable for the original state at $\Delta = 0$ ($\beta = 0$). Under the perturbation $\Delta$ ($0 \le \beta \le \beta_{\max}$), the norm of transfer function $G(\lambda)$ attains the maximum at $\lambda = i0$ with $||G(i0)||_{L(\mathbf{C})} = |\langle c(\cdot), (-A)^{-1}b(\cdot)\rangle_0|$. So, if it is assumed that $||\Delta||||G(i\omega)|| \le |\beta|||G(0)|| < 1$, then we have $0 \le |\beta| < r(A : D, E)$. Therefore, applying small gain theorem, we can say that the input-output stability is sustained if and only if $||\Delta|| < r(A : D, E)$.

**3.2. The stability of the nonlinear poulation system**
As well as the linear population system, we attempt to evaluate the stability of nonlinear population system. Since we cannot define a transfer function toward the nonlinear population system, we are not able to evaluate the system's stability in this form. Therefore, we have to linearize this nonlinear population system. The first equation of nonlinear population system (equation (9)) is as below.

$$\frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} = -\{\mu(a) + \delta \cdot p(a,t)\}p(a,t). \tag{31}$$

Consider the age-dependent population density function $p(a,t) + \Delta p$ which is $\Delta p$ apart from the criterion of the age-dependent population density function $p(a,t)$. Substituting $p(a,t) + \Delta p$ for $p(a,t)$ in the equation above, we can obtain

$$\frac{\partial p(a,t) + \Delta p}{\partial a} + \frac{\partial p(a,t) + \Delta p}{\partial t} = - \{\mu(a) + \delta \cdot (p(a,t) + \Delta p)\}(p(a,t) + \Delta p) \tag{32}$$

Modifying this equation becomes,

$$\frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} + \frac{\partial \Delta p}{\partial a} + \frac{\partial \Delta p}{\partial t} = - \{\mu(a) + \delta \cdot p(a,t)\}p(a,t) - \{\mu(a) + 2\delta \cdot p(a,t)\}\Delta p - \delta \cdot \Delta p\Delta p. \tag{33}$$

If we take $\Delta p \ll p(a,t)$, then we can disregard $\delta \cdot \Delta p\Delta p$. Considering equation (1), we can obtain

$$\frac{\partial \Delta p}{\partial a} + \frac{\partial \Delta p}{\partial t} = -\{\mu(a) + 2\delta \cdot p(a,t)\}\Delta p. \tag{34}$$

Therefore, taking the criterion of a certain time $t = \bar{t}$, we can obtain the linearized poulation system as below.

$$\frac{\partial \Delta p}{\partial a} + \frac{\partial \Delta p}{\partial t} = -\{\mu(a) + 2\delta \cdot p(a, \bar{t})\}\Delta p. \quad (35)$$

Applying the mortal rate $\mu(a) + 2\delta \cdot p(a, \bar{t})$ to equation (29), we can obtain

$$ReG(j\omega) = \int_0^{a_m} c(a) \frac{1}{\omega} \sin \omega a e^{-\int_0^a \mu(\xi)+2\delta\cdot p(a,\bar{t})d\xi} da,$$
$$ImG(j\omega) = \int_0^{a_m} c(a) \left(\frac{-1}{\omega}\right) (1 - \cos \omega a) e^{-\int_0^a \mu(\xi)+2\delta\cdot p(a,\bar{t})d\xi} da. \quad (36)$$

## 4. Optimal control of age-dependent population system

### 4.1. Introduction of the performance index

In the optimal control, we at first introduce the performance index which is the criterion of optimal control. Although there are many choices in selection of the performance index, we introduce the one which describes the difference of both the actual distribution and the target distribution. We attempt to control the age-dependent population system (equation (1)) within the time interval $[0, T]$. So the performance index can be described as

$$J(\beta, T) = \int_0^T \{\int_0^{a_m} [p(a,t) - p^*(a)]^2 \, da\} dt. \quad (37)$$

The corresponding discrete form with the state vector $\mathbf{x}(t)$ in the system of equation (5) is

$$\begin{aligned} J(\beta, T) &= \sum_{t=0}^{T-1} \sum_{i=1}^{a_m} [x_i(t) - x_i^*]^2 \\ &= J = \sum_{t=t_0}^{T-1} [\mathbf{x}(t) - \mathbf{x}^*]^\tau [\mathbf{x}(t) - \mathbf{x}^*], \end{aligned} \quad (38)$$

where $a_m$ denotes the maximum age and $\tau$ indicates transposition of the vector, $\mathbf{x}^*$ the ideal population state(target state). Given the ideal population state $\mathbf{x}^*$, the problem of optimal control for the population system(equation (5)) is to find the optimal birth rate $\beta^*(t)$. In this case, we define the optimal birth rate $\beta^*(t)$ as the birth rate which minimizes the performance index (37) or (38) with respect to the admissible control $||\Delta|| < r(A : D, E)$. $\beta^*(t)$ satisfies

$$J(\beta^*(t), T) = \min_{\beta(t)\in\mathbf{U}} \int_0^T \{\int_0^{a_m} [p(a,t) - p^*(a)]^2 \, da\} dt, \quad (39)$$
$$J(\beta^*(t), T) = \min_{\beta(t)\in\mathbf{U}} \sum_{t=t_0}^{T-1} [\mathbf{x}(t) - \mathbf{x}^*]^\tau [\mathbf{x}(t) - \mathbf{x}^*]. \quad (40)$$

### 4.2. The method of calculating the optimal birth rate

This section is concerned with the method of calculating the optimal birth rate $\beta^*(t)$ which realizes the target population state $\mathbf{x}^*$. In this section, we use the discrete poulation system described as

$$\mathbf{x}(t+1) = \mathbf{H}(t)\mathbf{x}(t) + \beta(t)\mathbf{B}(t)\mathbf{x}(t) = F(t, \mathbf{x}(t), \beta(t)), \quad (41)$$
$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

And the performance index is written as

$$J = \sum_{t=t_0}^{T-1} [\mathbf{x}(t) - \mathbf{x}^*]^\tau [\mathbf{x}(t) - \mathbf{x}^*]. \quad (42)$$

$\mathbf{x_0}$ is the initial population state which is the demographic data of Japan-2000. This is shown in Fig.3.

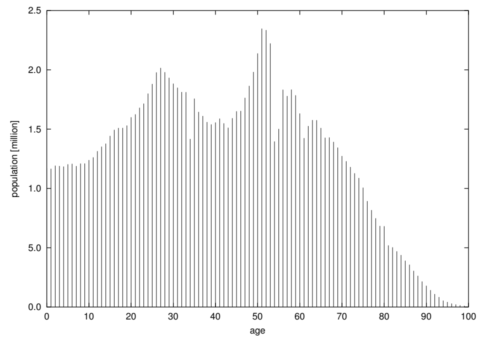


Fig. 3. Initial population state $\mathbf{x_0}$

$\mathbf{x}^*$ is the target population state and this is shown in Fig.4. If the model such as Fig.4 will be realized, then we will be able to maintain the distribution by choosing the appropriate birth rate. This is the reason why we chose this model for the target population state. In this model, the ratio of people aged $a(65 \le a)$ and people aged $a(15 \le a < 65)$ is set to be 1 to 4. This is the actual ratio of Japan-2000, and we chose this ratio for the target model.

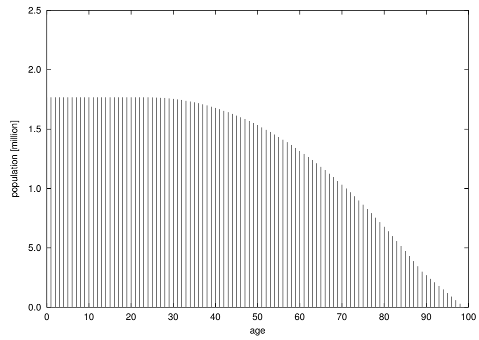


Fig. 4. Target population state $\mathbf{x}^*$

In the following, we shall apply the method of constrained gradient for calculation of the optimal birth rate. For this end, we need to calculate the gradient of the functional of the performance index. Define the Hamilton function as

$$H(t, \mathbf{x}(t), \beta(t), \psi(t+1)) = F_0(t, \mathbf{x}(t), \beta(t)), \\ +\psi^\tau(t+1)F(t, \mathbf{x}(t), \beta(t)) \quad (43)$$
$$F_0(t, \mathbf{x}(t), \beta(t)) = [\mathbf{x}(t) - \mathbf{x}^*]^\tau [\mathbf{x}(t) - \mathbf{x}^*], \quad (44)$$

where $\psi(t+1)$ is the adjoint function determined by the following equation

$$\psi(t) = \frac{\partial H(t, \mathbf{x}(t), \beta(t), \psi(t+1))}{\partial \mathbf{x}}. \quad (45)$$

Given an increment $\Delta\beta$ sufficiently small to $\beta(t)$ ($\beta + \Delta\beta \in \mathbf{U}$), the corresponding increment $\Delta\mathbf{x}(t)$ should be

$$\begin{aligned} \Delta\mathbf{x}(t+1) &= F(t, \mathbf{x} + \Delta\mathbf{x}, \beta + \Delta\beta) - F(t, \mathbf{x}, \beta), \\ \Delta\mathbf{x}(t_0) &= 0. \end{aligned} \quad (46)$$

In addition, the performance index can be written as

$$J=\sum_{t=t_0}^{T-1}\left[\mathbf{x}(t)-\mathbf{x}^*\right]^\tau\left[\mathbf{x}(t)-\mathbf{x}^*)\right]$$
$$=\sum_{t=t_0}^{T-1}\left[H(t,\mathbf{x}(t),\beta(t),\psi(t+1))-\psi^\tau(t+1)F(t,\mathbf{x}(t),\beta(t))\right].$$

Hence the increment $\Delta J$ of $J$ is

$$\begin{aligned}\Delta J &= J(\beta+\Delta\beta)-J(\beta)\\ &= \sum_{t=t_0}^{T-1}\left[H(t,\mathbf{x}+\Delta\mathbf{x},\beta+\Delta\beta,\psi)-H(t,\mathbf{x},\beta,\psi)\right]\\ &- \sum_{t=t_0}^{T-1}\psi^\tau(t+1)\Delta\mathbf{x}(t+1).\end{aligned}$$

Within the interior of the domain of control, it can be unfolded into a Taylor series

$$\begin{aligned}\Delta J &= \sum_{t=t_0}^{T-1}\{\left[\frac{\partial H(t,\mathbf{x}(t),\beta(t),\psi(t+1))}{\partial\mathbf{x}}\right]^\tau\Delta\mathbf{x}+\frac{\partial H}{\partial\beta}\Delta\beta\\ &+ \ldots-\sum_{t=t_0}^{T-1}\psi^\tau(t+1)\Delta\mathbf{x}(t+1)\}. \end{aligned}\tag{47}$$

Since $\Delta\mathbf{x}(t_0)=0$, we also have

$$\begin{aligned}&- \sum_{t=t_0}^{T-1}\psi^\tau(t+1)\Delta\mathbf{x}(t+1)\\ &= -\sum_{t=t_0+1}^{T}\psi^\tau(t)\Delta\mathbf{x}(t)\\ &= -\sum_{t=t_0}^{T-1}\psi^\tau(t)\Delta\mathbf{x}(t)-\psi^\tau(T)\Delta\mathbf{x}(T).\end{aligned}$$

Substituting the above expression into equation (47), we have

$$\Delta J=\sum_{t=t_0}^{T-1}\left[\frac{\partial H}{\partial\mathbf{x}}-\psi(t)\right]^\tau\Delta\mathbf{x}(t)+\frac{\partial H}{\partial\beta}\Delta\beta-\psi^\tau(T)\Delta\mathbf{x}(T)-\cdots\tag{48}$$

If we let

$$\begin{aligned}\psi(t) &= \frac{\partial H}{\partial\mathbf{x}}=\frac{\partial F_0}{\partial\mathbf{x}}+\left[\frac{\partial F}{\partial\mathbf{x}}\right]^\tau\psi(t+1),\\ \psi(T) &= 0,\end{aligned}\tag{49}$$

we then obtain

$$\Delta J=\sum_{t=t_0}^{T-1}\frac{\partial H}{\partial\beta}\Delta\beta+\cdots\tag{50}$$

If we take the first term of $\Delta J$ in its expansion, and define the gradient of $\Delta J$ as

$$\Delta J_\beta=\frac{\partial H(t,\mathbf{x}(t),\beta(t),\psi(t+1))}{\partial\beta},\tag{51}$$

then

$$\Delta J_\beta=\frac{\partial F_0}{\partial\beta}+\left[\frac{\partial F}{\partial\beta}\right]^\tau\psi(t+1).\tag{52}$$

Now we introduce the constraining operator defined below

$$\beta(t)=\begin{cases}\beta_0, & (\beta(t)\le\beta_0)\\ \beta(t), & (\beta_0<\beta(t)<\beta_1)\\ \beta_1, & (\beta(t)\ge\beta_1)\end{cases}\tag{53}$$

In this paper, we used Penalty functions for the constraining operator as in [3] . We then obtain an iterative equation with constraints by the algorithm of steepest descent method.

$$\beta^{(k+1)}(t)=\beta^{(k)}(t)-\epsilon\{\Delta J_\beta^{(k)}+c(\beta^{(k)})\}.\tag{54}$$

$\epsilon$ is a positive constant and $c(\beta^{(k)})$ denotes the Penalty function. In this equation, by taking $k$ large enough, $\beta^{(k)}$ converges to $\beta^*(t)$.

## 5. Reslults

We could figure the vector locus of loop transfer function of both linear and linearized population system, plotting from $\omega=0$ to $\omega=\infty$. This figure is shown in Fig.5. As for the nonlinear population system, we made computation for various values of $\delta$. From these figures, we could say that all of these systems were stable when the birth rate $\beta(t)$ was appropriately chosen. We could also say that these systems were also stable for each $\delta$.

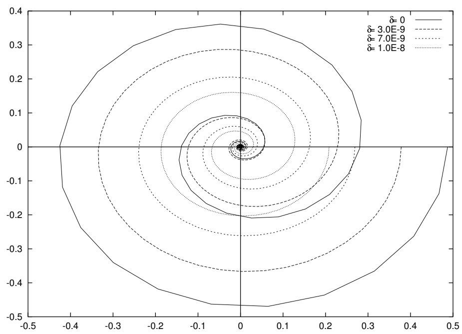


Fig. 5. Comparison of the vector locus transfer function

Additionally, we could obtain the values of stability radius $r(A:D;E)$ corresponding to each $\delta$. This is shown in Table 1.

Table 1. The values of $r(A:D;E)$ corresponding to each $\delta$

| $\delta$ | 0 | $1.0\times10^{-9}$ | $3.0\times10^{-9}$ |
|---|---|---|---|
| $\|\|G(i0)\|\|$ | 0.48 | 0.438 | 0.355 |
| $r(A:D;E)$ | 2.052 | 2.282 | 2.819 |

| $\delta$ | $5.0\times10^{-9}$ | $7.0\times10^{-9}$ | $1.0\times10^{-8}$ |
|---|---|---|---|
| $\|\|G(i0)\|\|$ | 0.287 | 0.233 | 0.171 |
| $r(A:D;E)$ | 3.479 | 4.289 | 5.860 |

Secondly, we computed and figured the optimal birth rates $\beta^*(t)$ corresponding to each $\delta$. In the process of population control, we computed $\beta^*(t)$ in the case of $T=50$ and $T=100$, where $T$ denotes the interval of control action. These results are shown in Fig.6 and Fig.7.

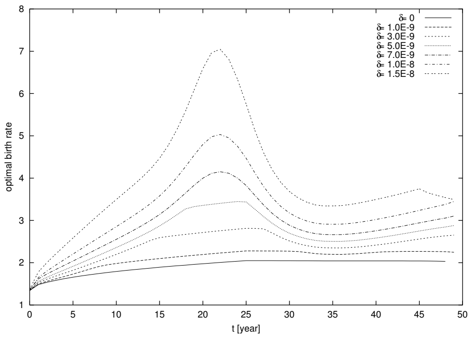


Fig. 6. The optimal birth rate $\beta^*(t)$ $(T = 50)$

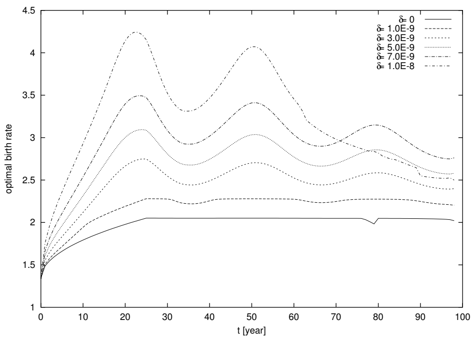


Fig. 7. The optimal birth rate $\beta^*(t)$ $(T = 100)$

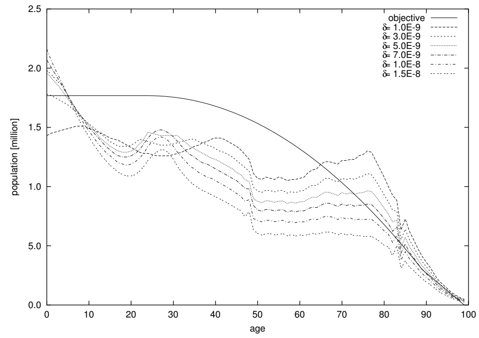


Fig. 8. The population state $\mathbf{x}(T)$ $(T = 50)$

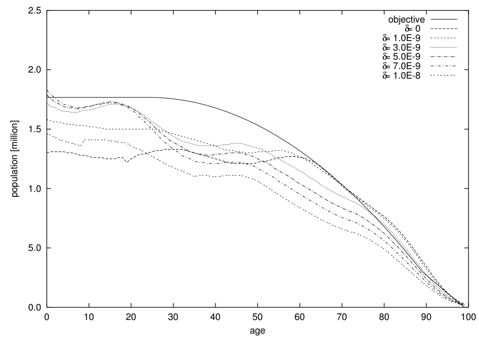


Fig. 9. The population state $\mathbf{x}(T)$ $(T = 100)$

In addition, we figured the population state at the year $t = T$ which was the result of calculation, depending on the calculated $\beta^*(t)$. From the obtained figures, we couldn't realize the target distribution when calculated in the case of $T = 50$, especially when we took $\delta$ small enough. On the other hand, we could obtain the figures which were close to the target distributions, when calculated in the case of $T = 100$. These figures are shown in Fig.8 and Fig.9. It is natural that such results were obtained, because the time constant $\tau$ of this age-dependent population system is very large. The time constant $\tau$ equals to the life expectancy of the age-dependent population system. This fact was proved in [1].

## 6. Conclusions

In this study, we could obtain the stability radius $r(A : D; E)$ of both linear and linearized population system by figuring the vector locus of loop transfer function. From these results, we could conclude that the higher values we took for $\delta$, the higher the stability radius $r(A : D; E)$ became. On the other hand, the smaller values we took for $\delta$, the smaller the stability radius $r(A : D; E)$ became. From this result, we can conclude that the higher we took the values of $\delta$, the stronger the stability of the population system became. Secondly, we could determine the optimal birth rates $\beta^*(t)$ that realized a target distribution when we took the interval of control action $T$ higher than the time constant $\tau$ which is actually the life expectancy. But in case of taking the interval of control action $T$ smaller than the time constant $\tau$, we could not realize a target distribution. For further studying, we might have to use the population system taking immigrants into consideration.

## References

[1] Song Jian and Yu Jingyun, "Population system control ," *Springer-Verlag. NewYork*, 1980.

[2] N.Kunimatsu, "Robust stability of population dynamical systems," unpublished.

[3] L.C.W.Dixon, "Nonlinear optimal computating," *Baihukan. Japan*, 1974.

[4] W.L.Chan and G.B.Zhu, "Optimal birth control of population dynamics," *J. Math. Anal. Appl.*, vol. 144, pp. 532-552, 1980.