The Guaranteed Bound of Horizon Size for the Stabilizing Receding Horizon Control

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Abstract: In this paper, we derive the guaranteed bound of the horizon size for the stabilizing receding horizon control(RHC). From the convergence property of the solution to the Riccati equation, it is shown that the lower bound can be represented in terms of the parameters in the given system model, which makes an off-line calculation possible. Additionally, it is shown to be able to obtain the stabilizing RHC without respect to the final weighting matrix. The proposed guaranteed bound is obtained numerically via simulation.

Keywords: Guaranteed bound, horizon, stabilizing receding horizon control, Riccati equation,

1. Introduction

Receding horizon control(RHC), also known as model predictive control(MPC), has received much attention due to its many advantages such as good tracking performance, I/O constraints handling and extension to the nonlinear system compared with the steady state linear quadratic(LQ) control[1], [2], [3].

In order to obtain a stabilizing RHC, the final weighting matrix is required to satisfy some matrix inequality [4]. If control gains of the RHC are obtained from these final weighting matrices, the stability is guaranteed irrespective of the horizon size. If the system dimension is large, it may be difficult to find out a proper final weighting matrix from the matrix inequality. In this paper, the stability based on the horizon size is considered regardless of a final weighting matrix. We provide a guide on how large the horizon size should be for a stabilizing RHC. In this paper, it is not necessary to find out a proper final weighting matrix separately. Additionally, the guaranteed horizon is represented in terms of the parameters in the given system model, which makes an off-line calculation possible.

This paper is structured as follows: Section 2 introduces some preliminaries of the RHC; Section 3 describes the derivation of the guaranteed bound of the horizon size for the stabilizing RHC, Section 4 provides the numerical example and finally, the conclusions are stated in Section 5.

2. Preliminaries of the RHC

Let consider the following discrete time-invariant linear system without constraints:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{1}$$

where $u_k \in R^n$ is the control input and $x_k \in R^m$ is the state of the plant, and $y_k \in R^p$ is the output of the plant. The optimal control is obtained first on the horizon $[k, k+N]$. Here k indicates the current time and N is the horizon size, i.e., $k+N$ is the final time.

The cost function is given by

$$\begin{aligned} J(x_k, k, k+N) &= \min_u \sum_{i=0}^{N-1} \frac{1}{2}(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}) \\ &+ \frac{1}{2} x_{k+N}^T Q_{k+N} x_{k+N} \end{aligned} \tag{2}$$

where $Q \geq 0$, $R > 0$, and $Q_{k+N} \geq 0$. Then, the optimal solution on the the horizon $[k, k+N]$ can be obtained from

$$\begin{aligned} u_{k+i}^* &= -[R + B^T P_{k+i+1,k+N} B]^{-1} B^T P_{k+i+1,k+N} A x_{k+i} \\ & i=0, ..., N-2, N-1 \end{aligned} \tag{3}$$

where the Riccati equation are given by

$$\begin{aligned} P_{k+i,k+N} &= A^T[I + P_{k+i+1,k+N} B R^{-1} B^T]^{-1} P_{k+i+1,k+N} A + Q \\ & i=N-1, N-2, ..., 1 \end{aligned} \tag{4}$$

with

$$P_{k+N,k+N} = Q_{k+N} \tag{5}$$

The receding horizon LQ control at time k is given by the first control u_k, which can be obtained from (3) with $i = 0$ as

$$u_k^* = -[R + B^T P_{k+1,k+N} B]^{-1} B^T P_{k+1,k+N} A x_k \tag{6}$$

From (6), it can be seen that the RDE(4) should be iteratively computed $N-1$ times with (5) to obtain the RHC. In order to obtain a stabilizing RHC, it is necessary to determine an appropriate horizon size to obtain the steady state solution of RDE.

3. Guaranteed bound of the Horizon Size

In this section, we derive a numerical guaranteed bound of the horizon size for the stabilizing RHC.

Let

$$R_{0,k+1} \triangleq R + B^T P_{k+1,k+N} B \tag{7}$$

$$L_k \triangleq [R + B^T P_{k+1,k+N} B]^{-1} B^T P_{k+1,k+N} A \tag{8}$$

$$A_{c,k} \triangleq A - BL_k \tag{9}$$

$$\tilde{P}_{k+1,k+N} \triangleq P_{k+1,k+N} - P \tag{10}$$

and R_0, L and A_c is the steady state value of $R_{0,k+1}$,L_k and $A_{c,k}$, respectively.
Then, from the convergence property of the solution of RDE, we have

$$R_{0,k+1} - R_0 = B^T \tilde{P}_{k+1,k+N} B \tag{11}$$
$$L_k - L = R_{0,k+1}^{-1} B^T \tilde{P}_{k+1,k+N} A_c \tag{12}$$
$$A_{c,k} = (I - BR_{0,k+1}^{-1} B^T \tilde{P}_{k+1,k+N}) A_c \tag{13}$$

and

$$\begin{aligned}\tilde{P}_{k+i,k+N} &= A_c^T \tilde{P}_{k+i+1,k+N} A_c \\ &- A_c^T \tilde{P}_{k+i+1,k+N} B R_{2,k+i}^{-1} B^T \tilde{P}_{k+i+1,k+N} A_c\end{aligned} \tag{14}$$

From (14),

$$\tilde{P}_{k+i,k+N} \leq A_c^T \tilde{P}_{k+i+1,k+N} A_c \tag{15}$$

Since $\rho(A) \leq \|A\|_\rho$ for any matrix norm, where $\rho(A)$ is the spectral radius of a matrix A. Thus from (13) we have

$$\begin{aligned}\|A_{c,k}\|_\rho &\leq (\|I\|_\rho + \|BR_{0,k+1}^{-1} B^T\|_\rho \|\tilde{P}_{k+1,k+N}\|_\rho)\|A_c\|_\rho \\ &\leq (\|I\|_\rho + \|BR^{-1} B^T\|_\rho \|\tilde{P}_{k+1,k+N}\|_\rho)\|A_c\|_\rho\end{aligned} \tag{16}$$

and from (15) we have

$$\|\tilde{P}_{k+i,k+N}\|_\rho \leq \|A_c^T\|_\rho \|\tilde{P}_{k+i+1,k+N}\|_\rho \|A_c\|_\rho \tag{17}$$

From (17),

$$\|\tilde{P}_{k+1,k+N}\|_\rho \leq \|A_c^T\|_\rho^{N-1} \|\tilde{P}_{k+N,k+N}\|_\rho \|A_c\|_\rho^{N-1} \tag{18}$$

Substituting (18) into (16), we have

$$\begin{aligned}\|A_{c,k}\|_\rho &\leq (\|I\|_\rho + \|BR^{-1} B^T\|_\rho \|\tilde{P}_{k+1,k+N}\|_\rho)\|A_c\|_\rho \\ &\leq (\|I\|_\rho + \|BR^{-1} B^T\|_\rho \|A_c^T\|_\rho^{N-1} \\ &\times \|\tilde{P}_{k+N,k+N}\|_\rho \|A_c\|_\rho^{N-1})\|A_c\|_\rho\end{aligned} \tag{19}$$

For the stabilizing RHC, the spectral radius of the closed loop matrix A_c must be less than 1. Thus,

$$\begin{aligned}(\|I\|_\rho + &\|BR^{-1} B^T\|_\rho \|A_c^T\|_\rho^{N-1} \\ &\times \|\tilde{P}_{k+N,k+N}\|_\rho \|A_c\|_\rho^{N-1})\|A_c\|_\rho < 1\end{aligned} \tag{20}$$

From (20),

$$\|A_c^T\|_\rho^{N-1} \|\tilde{P}_{k+N,k+N}\|_\rho \|A_c\|_\rho^{N-1} < \frac{\frac{1}{\|A_c\|_\rho} - \|I\|_\rho}{\|BR^{-1}B^T\|_\rho} \tag{21}$$

Apply the natural logarithm to both sides of the equation (21) getting

$$(N-1)(\ln \|A_c^T\|_\rho + \ln \|A_c\|_\rho) < \ln \frac{\frac{1}{\|A_c\|_\rho} - \|I\|_\rho}{\|BR^{-1}B^T\|_\rho \|\tilde{P}_{k+N,k+N}\|_\rho}$$

Thus

$$N > 1 + \Xi \tag{22}$$

where

$$\Xi = \frac{\ln \frac{\frac{1}{\|A_c\|_\rho} - \|I\|_\rho}{\|BR^{-1}B^T\|_\rho \|\tilde{P}_{k+N,k+N}\|_\rho}}{\ln \|A_c^T\|_\rho + \ln \|A_c\|_\rho} \tag{23}$$

In order to obtain $\|A_c\|_\rho$ in (23), the eigenvalues of the closed loop matrix must be known. Fortunately, the eigenvalues can be obtained by computing those of the hamiltonian matrix. Let the Hamiltonian is as follows:

$$H(x_k, u_k, \lambda_{k+1}) = \frac{1}{2}(x_k^T Q x_k + u_k^T R u_k) + \lambda^T (A x_k + B u_k) \tag{24}$$

Thus we can obtain the following Hamiltonian matrix corresponding to the above Hamiltonian:

$$\mathbf{H} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix} \tag{25}$$

That is,

$$\begin{bmatrix} x_{i+1} \\ \lambda_i \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_i \\ \lambda_{i+1} \end{bmatrix} \tag{26}$$

Under the assumption that (A,B) is controllable and A is nonsingular, (26) can be rewritten as follows:

$$\begin{bmatrix} x_{i+1} \\ \lambda_{i+1} \end{bmatrix} = \mathbf{H}_F \begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} \tag{27}$$

where

$$H_F = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix} \tag{28}$$

or

$$\begin{bmatrix} x_i \\ \lambda_i \end{bmatrix} = \mathbf{H}_B \begin{bmatrix} x_{i+1} \\ \lambda_{i+1} \end{bmatrix} \tag{29}$$

where

$$H_B = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^T \\ QA^{-1} A^T + & QA^{-1}BR^{-1}B^T \end{bmatrix} \tag{30}$$

The first n eigenvalues of H_F are the eigenvalues of $A_c = A - BL$ and are all inside the unit circle, $|\lambda_i| < 1$, i.e., asymptotically stable.
The remaining eigenvalues of H_F satisfy:

$$\lambda_{n+i} = \frac{1}{\lambda_i} \qquad i = 1, ..., n \tag{31}$$

Thus we can obtain

$$\|A_c\|_\rho = \max\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\} \tag{32}$$

In this paper, we consider the case of $Q_{k+N} = 0$, i.e., $P_{k+N,k+N} = 0$. Thus, $\|\tilde{P}_{k+N,k+N}\|_\rho = \|P\|_\rho$. There some results for the guaranteed bound for the solution of Riccati equation[5], [6], [7], [8]. In [6], [7],the following guaranteed bound of the solution of RDE is given:

$$P \geq A^T (P_0^{-1} + BB^T)^{-1} A + Q \tag{33}$$

where the positive definite matrix P_0 is defined as

$$P_0 \geq A^T (\xi^{-1} I + BB^T)^{-1} A + Q \tag{34}$$

with

$$\begin{aligned}\eta &= \sigma_n^2(A) + \sigma_1^2(B)\lambda_n(Q) - 1 \\ \xi &= \frac{\eta + [\eta^2 + 4\sigma_1^2(B)\lambda_n(Q)]^{\frac{1}{2}}}{2\sigma_1^2(B)}\end{aligned} \tag{35}$$

In (35), $\lambda_i(A)$ is the *ith* eigenvalue of A , all of $|\lambda_i(A)|$ are arranged in non-increasing order; $\sigma_i(A)$ is the *ith* singular value of A, the values of $\sigma_i(A)$ are arranged in non-increasing order.

From (32)-(35), we can obtain the spectral radius of the closed loop matrix and the bound of the solution of RDE so that (22) and (23) can be rewritten as follows:

$$N > 1 + \frac{\ln \frac{\frac{1}{\rho(A_c)} - 1}{\rho(BR^{-1}B^T)\rho(\hat{P})}}{2 \ln \rho(A_c)} \tag{36}$$

where $\hat{P}$ is the guaranteed bound of P obtained from (33). Thus the guaranteed bound of the horizon size for the stabilizing RHC can be obtained through off-line calculation from (36). Furthermore, the stabilizing RHC can be obtained without respect to the final weighting matrix if the horizon size is larger than the guaranteed bound due to the restriction of the spectral radius of the closed loop matrix to be less than 1 in the derivation of the guaranteed bound.

4. Simulation

In this section, a numerical example is given with the following system matrices:

$$A = \begin{bmatrix} 2 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{37}$$

and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad Q_N = 0. \tag{38}$$

The initial state is $x_0 = [-2, 3]$.

The obtained horizon size by the trial and error is $N = 10$. From the $Fig.1$ and $Fig.2$, it can be seen that the closed loop system is unstable when $N = 9$ but stable when $N = 10$.

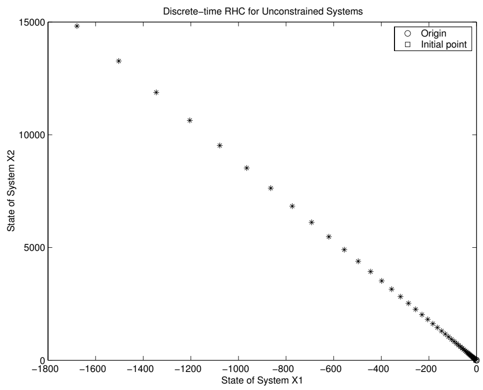

Fig. 1. RHC for Horizon Size N=9(State)

From (36), we can obtain the guaranteed bound of the horizon size as $N = 25$ for this example. The obtained guaranteed bound is larger than the horizon size obtained by trial, thus it can be noted that the stabilizing RHC can be obtained if the horizon size is larger than 25 considering the stabilizing RHC is obtained with horizon size of 10.

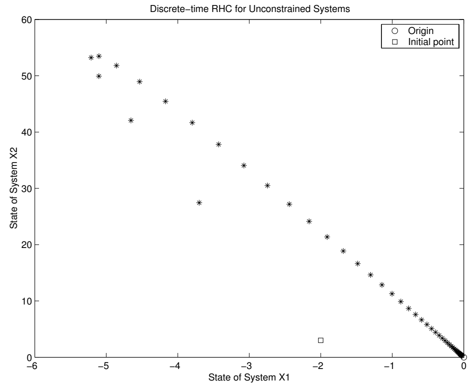

Fig. 2. RHC for Horizon Size N=10(State)

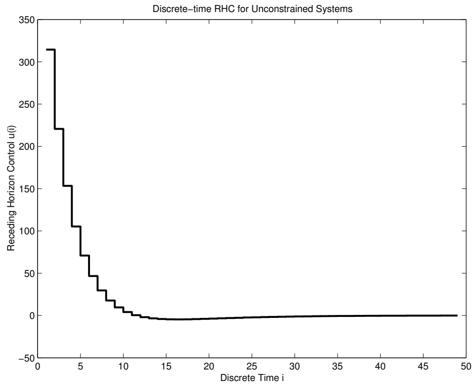

Fig. 3. RHC for Horizon Size N=10(Control)

5. Conclusions

In this paper, the guaranteed bound of the horizon size for the stabilizing RHC is derived, which can be obtained by off-line calculation due to its representation in terms of the parameters in the given system model. Because we considered the case of zero final weighting matrix, thus the obtained RHC is easier to be stabilizing for any positive final weighting matrix when the horizon size is larger than the guaranteed bound. In other words, the stabilizing RHC can be obtained without respect to the final weighting matrix. The proposed guaranteed bound is obtained numerically via simulation.

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