Discretization Behaviors of Equivalent Control Based Sliding-Mode Systems

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Abstract: In this paper, we study the discretization behaviors of the equivalent control based sliding-mode control (SMC) systems. The investigation is carried out via studying the second order linear system and some interesting dynamic properties are explored. Especially, according to the variation of system parameters, the inherent dynamical behavior of the trajectories within some specified boundaries are studied. From this result, the boundaries of system state using symbolic dynamics approach are first proposed. Simulations are presented to verify the theoretical results.

Keywords: Discretization behaviors, Steady state behaviors, Sliding-mode control, The equivalent control

1. INTRODUCTION

Almost all of studies of sliding mode control (SMC) have been proposed in the continuous-time domain. In the actual system, however, controller is implemented in the discrete-time domain since they use digital computer. Therefore, discrete time sliding mode control (DSMC) has been studied extensively to address the problems in particular associated with the SMC of discrete time systems with relatively low switching. Major research efforts in DSMC have been devoted to the development of various controllers using specific guiding principles [1]-[3]. However, the study of discretizing continuous SMC for discrete implementation has not been fully explored [4]. Recently the chaotic and discrete behaviors of the equivalent control based SMC is attracted by discretizing continuous time SMC for several class of SMC system [5].

In this paper, we study the discretization of the equivalent control based sliding-mode control (SMC) systems. We first investigate some inherent dynamical properties of the discretization effect on the continuous time SMC and periodic properties which is changed by the variation of system parameter during the steady state phase. From this result, the behavior and boundary for the system steady states using symbolic dynamics approach are explored.

The investigation is mainly carried out via studying the second order linear systems and some interesting characteristics are explored.

2. DISCRETIZATION OF AN EQUIVALENT CONTROL BASED SMC

The controllable single-input linear Sliding Mode Control (SMC) and switching manifold is then described by

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

$$g = c^T x(t) \tag{2}$$

where $x \in R^n$ is the state vector, $u \in R^n$ is the scalar input, and A is an $n \times n$ matrix, b and c are $n \times m$ and $m \times n$ matrics respectively. The switching surface g is predefined to represent a desired asymptotically stable dynamics. From the above system, the equivalent control input based SMC can be obtained as

$$u = u_{eq} + u_s \tag{3}$$

where $u_{eq} = -(c^T b)^{-1} c^T Ax$ and $u_s = -\alpha (c^T b)^{-1} \operatorname{sgn} g(x)$ with $\alpha > 0$ being a constant control gain, sgn is the sign function, and $c^T b$ is nonsingular. Note that the equivalent control u_{eq} is derived by solving $\dot{g} = 0$ subject to (1).

Consider the concept of an equivalent discrete-time model motivated by eventual digital computer implementations of algorithm. Also our present interest in this study is how discretization affects the control performance of this class of SMC. Thus, we obtain discrete-time measurements from a continuous time system described by (1), with u held constant over each sample period from sample time $[kh,\ (k+1)h]$, where h is a sampling period.

To study the discretization behaviors, we first convert the continuous-time system (1) under the ZOH into the discrete form. At the discrete time k , the solution can be written as

$$x(k+1) = e^{Ah} x(k) + \int_0^h e^{A\tau} b u_k \tag{4}$$

where

$$\begin{aligned} u_k &= u_{eq} + u_s \\ &= -c^T Ax(k) - \alpha \operatorname{sgn} g(x(k)), \quad k = 0, 1, 2, \cdots. \end{aligned}$$

The Function forms a sequence of binary values of -1 and $+1$, which can be considered as a symbolic sequence of the dynamics. For simplicity, we denote $\operatorname{sgn} g(x(k))$ as s_k , hence the symbolic sequence, denoted as s , can be represented by $s = (s_0, s_1, s_2 \cdots)$. If a symbolic sequence has a minimal period L , we name the sequence a $period - L\ sequence$. The above discrete system can then be rewritten as

$$x(k+1) = \Phi x(k) - \alpha \Gamma s_k \tag{5}$$

with $\Phi = e^{Ah} - \int_0^h e^{A\tau} d\tau b c^T A$ and $\Gamma = \int_0^h e^{A\tau} d\tau b$.

It is well known that discretizing SMC with moderate sampling rates causes chattering. The problem of interest is how the control performance deteriorates when the sampling period increase.

3. DISCRETIZATION ANALYSIS: SECOND-ORDER SYSTEMS

To study of the discretization behaviors of the equivalent control based sliding-mode control systems, we assume that the controllable second order system are in the form of

$$A=\begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix},\ b=\begin{pmatrix} 0 \\ 1 \end{pmatrix},\quad c=\left[c_1,\ 1\right]^T,\ c_1>0\ .$$

For this system,

$$\phi=e^{Ah}=e^{\beta h}\begin{pmatrix} \cos\zeta h+\beta\zeta^{-1}\sin\zeta h & \zeta^{-1}\sin\zeta h \\ -a_1\zeta^{-1}\sin\zeta h & \cos\zeta h-\beta\zeta^{-1}\sin\zeta h \end{pmatrix}$$

with $\beta=a_2/2$, $\zeta=(1/2)\sqrt{4a_1-a_2^2}$. Therefore, it can be verified via some calculations that $\Phi=\begin{pmatrix} 1 & \upsilon \\ 0 & d \end{pmatrix},\ \Gamma=\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ where

$$\begin{aligned} v(h)&=e^{-\beta h}\zeta^{-1}\sin\zeta h-(c-a_2)\int_0^h \zeta^{-1}e^{-\beta\tau}\sin\zeta\tau d\tau \\ d(h)&=e^{-\beta h}(\cos\zeta h+(\beta-c)\zeta^{-1}\sin\zeta h) \\ \gamma_1(h)&=\int_0^h \zeta^{-1}e^{-\beta\tau}\sin\zeta\tau d\tau \\ \gamma_2(h)&=\zeta^{-1}e^{-\beta\tau}\sin\zeta h\ . \end{aligned}$$

Using the above equations, the second-order discrete time dynamics are then described by

$$x_1(k+1)=x_1(k)+vz(k)-\gamma_1\alpha s_k \tag{6}$$
$$z(k+1)=dz(k)-\gamma_2\alpha s_k \tag{7}$$

where $z(k)=x_2(k)$ is a scalar variable.

Theorem 1: The system (6) and (7) is stable in the sense of Lyapunov if [5] :

$$-1<d<1\ . \tag{8}$$

Furthermore, the system state is bounded by

$$|x_1(\infty)|\le\gamma_1\alpha+\frac{(c_1^{-1}-\nu)|\gamma_2|\alpha}{1-|d|},\ \ |z(\infty)|\le\frac{|\gamma_2|\alpha}{1-|d|}\ . \tag{9}$$

Proof: First, the boundary of z will be verified below. Iterating z by $k+1$ times from the initial state $z(0)$ leads to $z(k+1)=d^{k+1}z(0)-\gamma_2\alpha\sum_{i=0}^{k}d^i s_{k-i}$. Therefore, we obtain

$$\begin{aligned} |z(k+1)|&\le|d|^{k+1}|z(0)|+|\gamma_2|\alpha\sum_{i=0}^{k}|d|^i \\ &=|d|^{k+1}|z(0)|+\frac{|\gamma_2|\alpha(1-|d|^{k+1})}{(1-|d|)} \end{aligned} \tag{10}$$

It is easy to know that as $k\to\infty$, z becomes $|z(\infty)|\le\frac{|\gamma_2|\alpha}{1-|d|}$. Since z is eventually confined by (9), we now find the lower and upper bounds for x_1 . At the intersection point of the switching line and the equilibrium $|z(\infty)|=\frac{|\gamma_2|\alpha}{1-|d|}$, we have $x_1(k)=-c_1^{-1}z(k)=-c_1^{-1}\frac{|\gamma_2|\alpha}{1-|d|}$. Therefore, from (6) we can get the boundary of x_1 to be $\gamma_1\alpha+\frac{(c_1^{-1}-\nu)|\gamma_2|\alpha}{1-|d|}$. ■

Theorem 2: If the system (6) and (7) is stable and Period-L sequence can be exactly estimated, then the boundary of system state is given by

$$|x(\infty)|=\gamma_1\alpha+\sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|\ (c_1^{-1}-\upsilon)}{1+|d|^{L/2}}\ ,$$

$$|z(\infty)|=\sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|}{1+|d|^{L/2}}\ . \tag{11}$$

Proof: First, it is clear that iterating z by $k+1$ times from the initial state $z(0)$ leads to

$$\begin{aligned} z(k+1)&=dz(k)-\gamma_2\alpha s_k \\ &=d^k z(0)-d^{k-1}\gamma_2\alpha s_0-d^{k-2}\gamma_2\alpha s_1-\cdots \end{aligned} \tag{12}$$

where if $k\to\infty$ and $|d|<1$ then $d^k z(0)$ is neglected. Eq. (12) can be rewritten as

$$z(k+1)\simeq-\gamma_2\alpha(s_{k-1}+ds_{k-2}+d^2 s_{k-3}+d^3 s_{k-4}\cdots) \tag{13}$$

where, if the symbolic sequence is period-2, i.e., $s=(+1,\ -1)$ then the state trajectory will periodically converge to the two fixed points. Therefore, in order to divide the right -hand side of (13) into each periodic set, we can group by two terms.

$$\begin{aligned} s_{k-2}+ds_{k-2}&=(1-d) \\ d^2 s_{k-3}+d^3 s_{k-4}&=d^2(1-d) \\ d^4 s_{k-5}+d^5 s_{k-6}&=d^4(1-d) \\ &\vdots \end{aligned} \tag{14}$$

Substituting (14) into (13) yields

$$z(k+1)\simeq-\gamma_2\alpha(\,(1-d)+d^2(1-d)+d^4(1-d)+\cdots) \tag{15}$$

where, as $k\to\infty$, we obtain $z\to-\frac{\gamma_2\alpha(1-d)}{1-d^2}$. Therefore we can express the state boundary of z as

$$|z(\infty)| = \frac{|\gamma_2|\alpha}{1+|d|}. \quad (16)$$

And next, if the symbolic sequence is period-4, i.e., $s=(+1,\ +1,-1,\ -1)$ then the trajectory will periodically converge to the four fixed points. we can also bind the right -hand side of (13) by forth terms.

$$\begin{aligned} s_{k-1}+ds_{k-1}+d^2s_{k-2}+d^3s_{k-3} &= (1-d^2)+\ d(1-d^2) \\ d^4s_{k-4}+d^5s_{k-5}+d^6s_{k-6}+d^7s_{k-7} &= d^4(1-d^2)+d^5(1-d^2) \quad (17) \\ &\vdots \end{aligned}$$

Substituting (17) into (13) yields

$$z(k+1) \simeq -\gamma_2\alpha((1-d^2)+d(1-d^2)+d^4(1-d^2)+\cdots) \quad (18)$$

where, as $k\to\infty$, we obtain

$$z \to -\frac{\gamma_2\alpha(1-d^2)}{1-d^4}-\frac{\gamma_2\alpha d(1-d^2)}{1-d^4}. \quad (19)$$

Therefore, we can also obtain the boundary of z

$$|z(\infty)| = \sum_{n=0}^{n=1}\frac{|\gamma_2|\alpha|d|^n}{1+|d|^2}\quad. \quad (20)$$

Consequently, from (16) and (20), it can be made more general form, which can be expressed as

$$|z(\infty)| = \sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\alpha|d|^n}{1-|d|^{L/2}}$$

Similarly, the boundary of x_1 can be derived as.

$$|x(\infty)| = \gamma_1\alpha+\sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|\ (c_1^{-1}-\upsilon)}{1+|d|^{L/2}}\quad. \qquad ■$$

Corollary 1: By (9) and (11), it is easily known that the relation between two equations can be defined as

$$\begin{aligned}|x(\infty)| &= \gamma_1\alpha+\sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|\ (c_1^{-1}-\upsilon)}{1+|d|^{L/2}} \\ &< \gamma_1\alpha+\frac{(c_1^{-1}-\upsilon)|\gamma_2|\ \alpha}{1-|d|}\ ,\end{aligned}$$

$$|z(\infty)| = \sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|}{1+|d|^{L/2}} < \frac{|\gamma_2|\alpha}{1-|d|}. \quad (21)$$

Proof : In (11), as $L\to\infty$, $|d|^{L/2}\to 0$, then

$$\begin{aligned}|z(\infty)| &= \sum_{n=0}^{(L/2-1)}\frac{|\gamma_2|\ \alpha\ |d^n|}{1+|d|^{L/2}} \\ &= \frac{|\gamma_2|\alpha+|\gamma_2|\alpha|d|+|\gamma_2|\alpha|d^2|+\cdots}{1+|d|^{L/2}} \quad (22) \\ &= \frac{|\gamma_2|\alpha}{1+|d|} < \frac{|\gamma_2|\alpha}{1-|d|}\ .\end{aligned}$$

From above results it can be seen that there is less the boundary of (11) by Theorem 2 than the boundary of (9) by Theorem 1. Although (11) is shown as excellent results, it is necessary to estimate the exact period-L. For this reason, the estimation methods of period-L will be introduced in the next Theorem.

Corollary 2: For the system (6) and (7) with the period-2, the lower and upper bound of z are $z_l = -\frac{|\gamma_2|\ \alpha}{1-|d|}$, $z_u = \frac{|\gamma_2|\ \alpha}{1-|d|}$ respectively. The state trajectory will periodically converges to the two fixed points $(z_i,\ z_u)$ during the steady state phase.

Proof : Note that for any fixed s_k (s_k is either $+1$ or -1) for $k=0,\ 1,\ 2,\ \cdots$, form (7), we have, as $k\to\infty$, $z_l = -\frac{\gamma_2\ \alpha}{(1-d)}s_k$. For this reason the upper and lower boundary $(z_i,\ z_u)$ can be defined as $(-\frac{|\gamma_2|\ \alpha}{1-|d|},\ \frac{|\gamma_2|\ \alpha}{1-|d|})$. Finally, from this discussion it can be found that z_u is equal to the $-z_l$. ■

Theorem 3 : In (6) and (7), if $\gamma_2>0$ then the system has the period-2 for symbolic sequence. On the other hand, if $\gamma_2<0$ then the system has period-L more than $L=4$.

Proof : As above mentioned, when the symbolic sequence is period-2, using $(z_i,\ z_u)$ which has been defined in Corollary 2, we can rewritten (7) as

$$z(k+1) = z_l(k) = dz_u(k)-\gamma_2\alpha s_k \quad (23)$$

then, since $z_u = -z_l$

$$-z_u(k) = dz_u(k)-\gamma_2\alpha s_k \quad (24)$$

Thus, (24) becomes as

$$z_u = \frac{\gamma_2\ \alpha}{1-d} \quad (25)$$

As can be seen in (25), because z_u is a positive value, γ_2 must have a positive value by the proof of Corollary 2. But if $\gamma_2<0$, then (24) cannot be satisfied. This means that the sign of γ_2 to be period-2 must be positive. And it should be noted that period-L has actually has more than $L=4$ for $\gamma_2<0$. Consequently the system parameter $d,\ \gamma_2$ for various sampling time h is shown as Fig. 1. In Fig 1, the system has period-2 for the sampling time to be $\gamma_2>0$. ■

4. SIMULATION STUDIES

In this section, we present some simulation studies to verify the theoretical results in the previous sections.

We first set $a_1=5,\ a_2=-2,\ c_1=c_2=1$ for simulation condition. This setting corresponds to a stable continuous time system. The sampling time and the initial condition are $h = 0.3$, $x(0) = (2.0, 2.0)$. First, the theoretical values of the boundaries given by Theorem 1 are $|x_1(\infty)| < 0.359$ and $|z(\infty)| < 0.588$. Since $\gamma_2 = 0.381 > 0$, we can see from Fig. 2 that it has the period-2. Therefore, proposed boundaries by Theorem 2 is $|x_1(\infty)| \le 0.319$ and $|z(\infty)| \le 0.282$. Also the boundary of z becomes $(-0.282, 0.282)$ during the steady state phase. We can easily verify that the two fixed points are within the boundaries given in Theorem 2 and eventually the trajectory converges to these points.

To look at another interesting phenomenon, we set $a_1 = -100,\ a_2 = 0,\ c_1 = c_2 = 1$ for simulation condition. Let $h = 0.35$ and $x(0) = (0.5, 0.5)$. And then phase portrait of system state is Fig. 3. First, the theoretical values of the boundaries given by Theorem 1 are $|x_1(\infty)| < 0.394$ and $|z(\infty)| < 0.356$. Since $\gamma_2 = -0.035 < 0$, we can see from Fig. 3 that it has the period-4. Therefore proposed boundaries by Theorem 2 is $|x_1(\infty)| \le 0.078$ and $|z(\infty)| \le 0.037$, which are demonstrated in the simulation. We can also verify that the four fixed points are within the boundaries given in Theorem 2 and eventually the trajectory converges to these points.

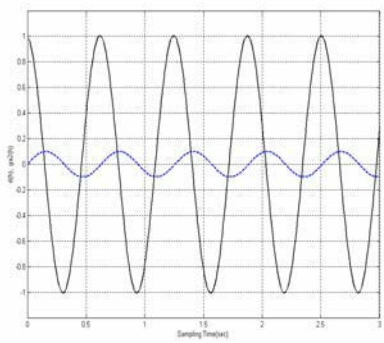

Fig. 1. System Parameter $d(h),\ \gamma_2(h)$

Dashed line: $d(h)$, Solid line: $\gamma_2(h)$

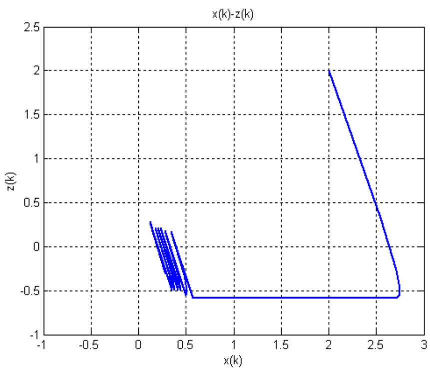

Fig. 2. Phase portrait of $h = 0.30$

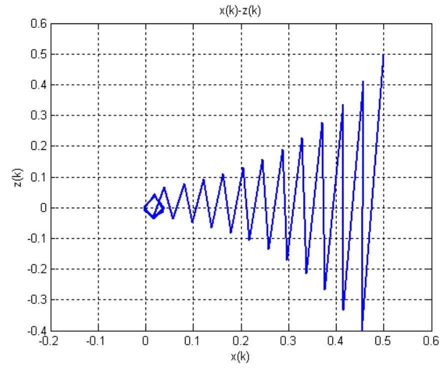

Fig. 3. Phase portrait of $h = 0.35$

5. CONCLUSION

In this paper, the dynamical properties of the discretized second-order systems has been presented. Especially, according to the variation of a sign of the system parameter γ_2, the estimation of period-L for symbolic sequence has been performed. From this result, for the case of $\gamma_2 > 0$, the behavior and boundary for the system states have been fully understood. The studies of these property will help prevent ill-behaviors due to discretization in digital controllers design.

Finally, further research will be devoted to investigate the estimation of period-L with $\gamma_2 < 0$

REFERENCES

[1] K. Furuta, “Sliding mode control of a discrete system,” *System & Control Letters*, vol 14, pp. 145-152, 1990.

[2] S. Z. Sarpturk, Y. Istefanopulos, and O. Kaynak, “On the stability of discrete-time skidding mode control system,” *IEEE Trans. on Automatic Control*, vol. 32, no. 10, pp. 930-932, 1987.

[3] P. Myszkorowski, “Robust controller for the linear discrete-time system,” *Proc. 30 th IEEE Conf. on Decision and Control*, England, pp. 2972-2973, 1990.

[4] X. Yu, “Discretizing Sliding Mode Control: From Chattering to Unstable Motion,” *Proc. 40 th IEEE Conf. on Decision and Control*, USA, pp. 914-918, 2001.

[5] X. Yu and G. Chen, "Discretization Behaviors of Equivalent Control Based Sliding-Mode Control Systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 9, pp.1641-1646, September 2003.

[6] L. O. Chua and T. Lin, “Chaos in digital filter,” *IEEE Trans. Circuits and Systems*, vol 35, pp. 648-652

[7] V. I. Utkin, "Variable Structure Systems with Sliding Modes," *IEEE Trans. on Automatic Control*, vol. AC-22, no. 2, pp.212-222, 1977.

[8] W. Gao, Y. Wang, and A. Homaifa, “Discrete-time variable structure control systems,” *IEEE Trans. Ind. Electron.*, vol 42, no.2, pp. 117-122, 1995.

[9] X. Yu and S. Yu , “Discrete sliding mode control design with invariant sliding sectors,” *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 122, no. 4, pp. 776-782, 2000.

[10] K. Furuta and Y. Pan, “Variable structure control with sliding sector,” *Automatica*, vol. 36, pp.211-228, 2000.