

A Note on Schur Stability of Real Weighted Diamond Polynomials

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Abstract: This paper presents a sufficient condition for the real weighted diamond polynomials to be Schur stable using bilinear transformation and Kharitonov’s theorem.

Keywords: Schur stability, Hurwitz stability, weighted diamond polynomial

1. Introduction

It is well known that Hurwitz stability of interval polynomials are equivalent to that of four special vertex polynomials[4]. Further, there is a simple example that the counterpart of Schur stability for interval polynomials does not hold. In those historical background, a sufficient conditions for interval polynomials to be Schur stable was studied by using the properties of bilinear transformation[3].

On the other hand, it was shown that the diamond polynomial is Hurwitz stable if and only if eight vertex polynomials are Hurwitz stable[1]. Similarly, there is a simple example that the counterpart of Schur stability for diamond polynomials does not hold. Further, Hurwitz stability conditions of a weighted diamond of polynomials have been studied[5],[6].

In this presentation, the properties of bilinear transformation matrices are firstly studied. And then, a sufficient conditions for a weighted diamond of polynomials to be Schur stable are studied by using the properties of bilinear transformation matrices and Kharitonov’s theorem. The present investigation is organized as follows. In Section 2, the bilinear transformation matrix will be given, and its properties would be investigated. In Section 3, the sufficient condition for real weighted diamond polynomials to be Schur stable will be presented. Section 4 will give an illustrative example. Finally, Section 5 will make some concluding remarks.

2. Bilinear Transformation Matrix and Its Properties

Consider an n -th degree real polynomial given by:

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, \quad (a_n \neq 0). \tag{1}$$

Applying the bilinear transformation:

$$z = \frac{s+1}{s-1}, \quad (s \neq 1),$$

to (1), we have

$$f(z) = \frac{1}{(s-1)^n} \sum_{i=0}^n a_n (s+1)^i (s-1)^{n-i}. \tag{2}$$

Define the numerator polynomial of (2) as

$$g(s) := b_n s^n + b_{n-1} s^{n-1} + \dots + b_0,$$

then the relation between the coefficients b_{n-i} ($i = 0, 1, \dots, n$) and a_i ($i = 0, 1, \dots, n$) can be written as

$$b_{n-i} = \sum_{j=0}^n p_{i,j}^n a_{n-j}, \quad (i = 0, 1, \dots, n), \tag{3}$$

where

$$p_{i,j}^n = \sum_{k=0}^i \binom{n-j}{k} \binom{j}{i-k} (-1)^{i-k},$$

$$\binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha - \beta)! \beta!}.$$

Using the following coefficient vectors and a matrix:

$$\mathbf{a}_n := [a_n, \dots, a_0]^T, \tag{4}$$

$$\mathbf{b}_n := [b_n, \dots, b_0]^T, \tag{5}$$

$$\mathbf{P}_n := [p_{i,j}^n]_{i,j=0,1,\dots,n} \in \mathbf{R}^{(n+1) \times (n+1)}, \tag{6}$$

it follows from (3)-(6) that

$$\mathbf{b}_n = \mathbf{P}_n \mathbf{a}_n.$$

The matrix \mathbf{P}_n is called Bilinear transformation matrix.

Proposition 1: In each row of the matrix \mathbf{P}_n , the element $p_{i,0}^n$ in the first column is maximum, *i.e.*,

$$p_{i,0}^n = \max_{0 \leq j \leq n} p_{i,j}^n, \quad i = 0, 1, \dots, n. \tag{7}$$

(Proof) See Appendix. ■

3. The Schur Stability of Real Weighted Diamond Polynomials

For an n -th degree real polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, \quad (a_n \neq 0),$$

consider the Schur stability of n -th degree weighted diamond polynomial:

$$D_n^{r,a^*,w}(z) = \{ f(z) = a_n z^n + \dots + a_0 \mid w_n |a_n - a_n^*| + \dots + w_0 |a_0 - a_0^*| \leq r \},$$

where $\{a_i^*\}_{i=0}^n$ are the given coefficients of an n -th degree real polynomial:

$$f^*(z) = a_n^* z^n + a_{n-1}^* z^{n-1} + \dots + a_0^*, \quad (a_n^* \neq 0),$$

and $\{w_i\}_{i=0}^n$ and r denote the positive weight and radius parameters which are given respectively.

We assume that the coefficient vector \mathbf{a}_n vibrates within the following weighted diamond domain:

$$\Gamma_a^n := \left\{ \mathbf{a}_n \in \mathbf{R}^{n+1} \mid w_n |a_n - a_n^*| + \dots + w_0 |a_0 - a_0^*| \leq r \right\}, \quad |a_n^*| > r/w_n. \quad (8)$$

Generally, the domain transformed by the bilinear transformation from a weighted diamond domain does not become weighted diamond domain, the transformation may cause distortion. Hence we derive the minimum surrounding box containing the transformed domain and study the Schur stability of real weighted diamond polynomials. Define the following coefficient vector and polynomials:

$$\begin{aligned} \mathbf{b}_n^* &:= [b_n^*, \dots, b_0^*]^T = \mathbf{P}_n \mathbf{a}_n^* = \mathbf{P}_n [a_n^*, \dots, a_0^*]^T, \\ g^*(s) &:= b_n^* s^n + b_{n-1}^* s^{n-1} + \dots + b_0^*, \\ g(s) &:= b_n s^n + b_{n-1} s^{n-1} + \dots + b_0, \end{aligned}$$

where $\{b_i\}_{i=0}^n$ is the vector components of (5). Moreover, defining the coefficients:

$$q_{n-i}^n := \max_{0 \leq j \leq n} \frac{|p_{n-i,j}^n|}{w_{n-j}} \geq 0, \quad i = 0, 1, \dots, n, \quad (9)$$

we have the following lemma.

Lemma 1: The following inequalities hold for each element b_i of the coefficient vector \mathbf{b}_n .

$$b_i^* - r q_{n-i}^n \leq b_i \leq b_i^* + r q_{n-i}^n, \quad i = 0, 1, \dots, n. \quad (10)$$

(Proof)

$$\begin{aligned} &|b_i - b_i^*| \\ &= | (p_{n-i,0}^n a_n + \dots + p_{n-i,n}^n a_0) \\ &\quad - (p_{n-i,0}^n a_n^* + \dots + p_{n-i,n}^n a_0^*) | \\ &= | p_{n-i,0}^n (a_n - a_n^*) + \dots + p_{n-i,n}^n (a_0 - a_0^*) | \\ &\leq | p_{n-i,0}^n (a_n - a_n^*) | + \dots + | p_{n-i,n}^n (a_0 - a_0^*) | \\ &\leq \frac{|p_{n-i,0}^n|}{w_n} w_n |a_n - a_n^*| + \dots + \frac{|p_{n-i,n}^n|}{w_0} w_0 |a_0 - a_0^*| \\ &\leq q_{n-i}^n w_n |a_n - a_n^*| + \dots \\ &\quad + q_{n-i}^n w_0 |a_0 - a_0^*|, \quad (\text{from (9)}) \\ &= q_{n-i}^n (w_n |a_n - a_n^*| + \dots + w_0 |a_0 - a_0^*|) \\ &\leq q_{n-i}^n r, \quad (\text{from (8)}) \end{aligned}$$

which imply (10). ■

Based on the above discussion, the real weighted diamond domain Γ_b^n derived by the bilinear transformation of Γ_a^n and its minimum surrounding box $\tilde{\Gamma}_b^n$ can be respectively defined as:

$$\begin{aligned} \Gamma_b^n &:= \left\{ \mathbf{b}_n \in \mathbf{R}^{n+1} \mid w_n \left| \sum_{j=0}^n p_{0j} b_{n-j} - 2^n a_n^* \right| + \dots \right. \\ &\quad \left. + w_0 \left| \sum_{j=0}^n p_{nj} b_{n-j} - 2^n a_0^* \right| \leq 2^n r \right\}, \\ \tilde{\Gamma}_b^n &:= \left\{ \tilde{\mathbf{b}}_n = [\tilde{b}_n, \dots, \tilde{b}_0]^T \in \mathbf{R}^{n+1} \mid \right. \\ &\quad \left. \alpha_i' \leq \tilde{b}_i \leq \beta_i', \quad (k = 0, 1, \dots, n) \right\}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \alpha_i' &:= b_i^* - r q_{n-i}^n, \\ \beta_i' &:= b_i^* + r q_{n-i}^n. \end{aligned}$$

Using the Kharitonov's polynomials:

$$\begin{aligned} k_1'(s) &= \alpha_n' s^n + \beta_{n-1}' s^{n-1} + \beta_{n-2}' s^{n-2} + \dots, \\ k_2'(s) &= \alpha_n' s^n + \alpha_{n-1}' s^{n-1} + \beta_{n-2}' s^{n-2} + \dots, \\ k_3'(s) &= \beta_n' s^n + \beta_{n-1}' s^{n-1} + \alpha_{n-2}' s^{n-2} + \dots, \\ k_4'(s) &= \beta_n' s^n + \alpha_{n-1}' s^{n-1} + \alpha_{n-2}' s^{n-2} + \dots, \end{aligned}$$

which correspond to the following interval polynomials

$$\begin{aligned} \tilde{f}(s) &= \tilde{b}_n s^n + \tilde{b}_{n-1} s^{n-1} + \dots + \tilde{b}_0, \\ \alpha_i' &\leq \tilde{b}_i \leq \beta_i', \quad i = 0, 1, \dots, n, \end{aligned} \quad (12)$$

we have the following theorem.

Theorem 1: If the Kharitonov's polynomials $k_i'(s)$ ($i = 1, 2, 3, 4$) are Hurwitz stable, then the real weighted diamond polynomial $D_n^{r,a^*,w}(z)$ is Schur stable.

(Proof) The proof follows from Lemma1, Kharitonov's theorem and the properties of bilinear transformation. ■

Besides, in the case of the diamond polynomials of which the weight w_i are 1 for any i , the maximum element in each row is always the element in the first column from Proposition 1. Therefore the computation of the maximum values in (9) is not necessary. Then the elements are determined easily as follows:

$$q_{n-i}^n = \max_{0 \leq j \leq n} \frac{|p_{n-i,j}^n|}{w_{n-j}} = p_{n-i,0}^n.$$

Using the parameters α_i'' and β_i'' defined by

$$\alpha_i'' := b_i^* - r p_{n-i,0}^n \quad \text{and} \quad \beta_i'' := b_i^* + r p_{n-i,0}^n,$$

and the Kharitonov's polynomials $k_i''(s)$ ($i = 1, 2, 3, 4$) given by replacing α_i' and β_i' in (12) as α_i'' and β_i'' , we have the following Corollary.

Corollary 1: If the Kharitonov's polynomials $k_i''(s)$ ($i = 1, 2, 3, 4$) are Hurwitz stable, then the real diamond polynomial $D_n^{r,a^*}(z)$ (with the weight one) is Schur stable. ■

4. An Example

As a simple example of real weighted diamond polynomials, consider the Schur stability of the following 2nd-degree real weighted diamond polynomial:

$$\begin{aligned} D_2^{6,a^*,w}(z) &= \{ f(z) = a_2 z^2 + a_1 z + a_0 \mid \\ &\quad 6|a_2 - 16| + 3|a_1 + 8| + 2|a_0 - 7| \leq 6 \}. \end{aligned}$$

In this case we have:

$$\begin{aligned} f^*(z) &= 16z^2 - 8z + 7, \\ g^*(s) &= 15s^2 + 18s + 31, \\ \Gamma_a^2 &= \left\{ \mathbf{a}_2 \in \mathbf{R}^3 \mid \right. \\ &\quad \left. 6|a_2 - 16| + 3|a_1 + 8| + 2|a_0 - 7| \leq 6 \right\}, \\ \Gamma_b^2 &= \left\{ \mathbf{b}_2 \in \mathbf{R}^3 \mid 3|b_2 + b_1 + b_0 - 64| \right. \\ &\quad \left. + 3|b_2 - b_0 + 16| + |b_2 - b_1 + b_0 - 28| \leq 12 \right\}, \end{aligned}$$

$$q_0^2 = \max\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right) = \frac{1}{2},$$

$$q_1^2 = \max\left(\frac{2}{6}, \frac{0}{3}, \frac{|-2|}{2}\right) = 1,$$

$$q_2^2 = \max\left(\frac{1}{6}, \frac{|-1|}{3}, \frac{1}{2}\right) = \frac{1}{2}.$$

Then the vectors $[\alpha'_2, \alpha'_1, \alpha'_0]^T$ and $[\beta'_2, \beta'_1, \beta'_0]^T$ can be respectively derived as follows:

$$\begin{bmatrix} \alpha'_2 \\ \alpha'_1 \\ \alpha'_0 \end{bmatrix} = \begin{bmatrix} 15 \\ 18 \\ 31 \end{bmatrix} - 6 \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} \beta'_2 \\ \beta'_1 \\ \beta'_0 \end{bmatrix} = \begin{bmatrix} 15 \\ 18 \\ 31 \end{bmatrix} + 6 \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 18 \\ 24 \\ 34 \end{bmatrix}$$

The domain \tilde{T}_b^2 can be written as (see Fig. 1):

$$\tilde{T}_b^2 = \{[\tilde{b}_2, \tilde{b}_1, \tilde{b}_0]^T \in \mathbf{R}^3 \mid \alpha'_i \leq \tilde{b}_i \leq \beta'_i, \quad (i = 0, 1, 2)\}.$$

Besides, it is easily checked that the Kharitonov polynomials $k'_1(s), k'_2(s), k'_3(s), k'_4(s)$ correspond to interval polynomials

$$g(s) = \tilde{b}_2 s^2 + \tilde{b}_1 s + \tilde{b}_0$$

in \tilde{T}_b^2 are Hurwitz stable. Therefore, the real weighted diamond polynomial $D_2^{6,a^*,w}(z)$ is Schur stable from Theorem 1. On the other hand, the actual Schur stable domain S_a^2 of the 2nd-degree polynomial $f(z) = a_2 z^2 + a_1 z + a_0$ is given by

$$S_a^2 = \{a_2 \in \mathbf{R}^3 \mid |a_0 + a_2| > |a_1|, |a_2| > |a_0|\}$$

Here, it is easily confirmed that the diamond domain Γ_a^2 is included in the domain S_a^2 . However it is very difficult to calculate the Schur stable domain S_a^n for arbitrary degree n . The proposed method works effectively for such case.

5. Conclusions

In this paper, a sufficient condition for the weighted diamond polynomials to be Schur stable was presented. The proposed condition was concluded as the Hurwitz stability of four Kharitonov's polynomials by calculating a minimum surrounding box containing the domain which was bilinear-transformed from an original weighted diamond domain. The necessary and sufficient condition for the real weighted diamond polynomials to be Schur stable should be studied as a future study.

References

[1] B. R. Barmish, R. Tempo, C. V. Hollot and H. I. Kang, "An Extreme point result for robust stability of a diamond of polynomials", *IEEE Transactions on Automatic Control*, Vol. AC-37, No.9, (1992), pp. 1460-1462.

[2] N. K. Bose, "Properties of the Q_n - matrix in bilinear transformation", *Proc. IEEE*, Vol.71, pp.1110-1111 (1983).

[3] N. K. Bose and E. Zeheb, "Kharitonov's theorem and stability test of multidimensional digital filters", *IEE proceedings*, Vol.133, Pt.G, No.4, pp.187-190 (1986).

[4] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations", *Differential Equations*, Vol. 14, (1979), pp. 1483-1485.

[5] V. L. Kharitonov and R. Tempo, "On the stability of a weighted diamond of real polynomials", *Systems & Control Letters*, Vol. 22, (1994), pp. 5-7.

[6] N. Otsuka and T. Okada, "On simplified Hurwitz stability conditions for low-degree weighted diamond polynomials", *Journal of The Franklin Institute*, Vol. 340, (2003), pp. 399-406.

APPENDIX

The proof of Proposition 1 in Section 2 is given. First we prepare the following two Lemmas to prove Proposition 1.

Lemma 2: [2]

- (i) $p_{0,j}^n = 1, \quad j = 0, 1, \dots, n,$
- (ii) $p_{n,j}^n = (-1)^j, \quad j = 0, 1, \dots, n. \quad \blacksquare$

Lemma 3:

- (i) $p_{i,0}^n = \binom{n}{i} \geq 0,$
 - (ii) $p_{i,j}^n = (-1)^i p_{i,n-j}^n, \quad i, j = 0, 1, \dots, n,$
 - (iii) $p_{i,j}^n = p_{i,j}^{n-1} + p_{i-1,j}^{n-1}, \quad i, j = 1, 2, \dots, n-1,$
- (Proof) Obviously (i) holds from the definition. Also (ii) holds from the following equalities:

$$\begin{aligned} (-1)^i p_{i,n-j}^n &= \sum_{k=0}^i \binom{j}{k} \binom{n-j}{i-k} (-1)^{-k} \\ &= \sum_{m=0}^i \binom{j}{i-m} \binom{n-j}{m} ((-1)^{-1})^{i-m} \\ &= p_{i,j}^n \end{aligned}$$

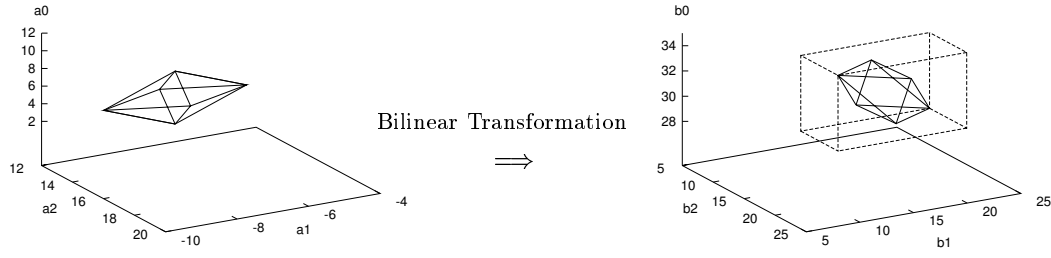
Now we prove (iii). For $\alpha, \beta \in \mathbf{N}$, the following equality holds and is well known as Pascal's triangular rule.

$$\binom{\alpha}{\beta-1} + \binom{\alpha}{\beta} = \binom{\alpha+1}{\beta}.$$

Hence, we have

$$\begin{aligned} p_{i,j}^{n-1} + p_{i-1,j}^{n-1} &= \sum_{k=0}^i \binom{n-1-j}{k} \binom{j}{i-k} (-1)^{i-k} \\ &\quad + \sum_{k=0}^{i-1} \binom{n-1-j}{k} \binom{j}{i-1-k} (-1)^{i-1-k} \\ &= \binom{n-j}{0} \binom{j}{i} (-1)^i \\ &\quad + \sum_{k=1}^i \binom{n-j}{k} \binom{j}{i-k} (-1)^{i-k} \\ &= \sum_{k=0}^i \binom{n-j}{k} \binom{j}{i-k} (-1)^{i-k} = p_{i,j}^n \end{aligned}$$

for $i, j = 1, 2, \dots, n-1. \quad \blacksquare$



(a) Weighted diamond domain Γ_a^2 (b) Transformed weighted diamond domain Γ_b^2 (solid line) and its surrounding box $\tilde{\Gamma}_b^2$ (broken line)

Fig. 1. Variation Domains of Coefficient Parameters

Using Lemmas 2 and 3, Proposition 1 can be proved as follows:

(Proof of Proposition 1) Obviously (7) holds from Lemma ??(i),(ii) and Lemma 3(i) for $i = 0$ and n . Next we prove

$$p_{i,0}^n = \max_{0 \leq j \leq n-1} p_{i,j}^n \quad (13)$$

for $i = 1, \dots, n-1$.

(i) In the case $n = 1, 2$: Obviously (13) holds from Definition.

(ii) In the case $n = 3, 4, \dots$: Assume that (13) holds when $n = k$. In this case we have

$$p_{i,0}^k \geq p_{i,j}^k, \quad i, j = 1, \dots, n-1. \quad (14)$$

On the other hand, it holds from (14) and Lemma ??(i) that

$$p_{i-1,0}^k \geq p_{i-1,j}^k, \quad i, j = 1, \dots, n-1. \quad (15)$$

Besides, the following equality can be derived from Lemma 3(iii):

$$\begin{aligned} p_{i,0}^{k+1} &= p_{i,0}^k + p_{i-1,0}^k, \\ p_{i,j}^{k+1} &= p_{i,j}^k + p_{i-1,j}^k. \end{aligned}$$

From the above equations, (14) and (15), we have

$$p_{i,0}^{k+1} \geq p_{i,j}^{k+1}, \quad i, j = 1, \dots, n-1.$$

Hence (13) holds for $n = k+1$. Therefore, (13) holds for any n by mathematical induction.

Besides, it follows from Lemma 3(i),(ii) that

$$p_{i,0}^n \geq p_{i,n}^n, \quad i = 1, \dots, n-1. \quad (16)$$

Thus, it is derived from (13) and (16) that

$$p_{i,0}^n = \max_{0 \leq j \leq n} p_{i,j}^n, \quad i = 1, \dots, n-1,$$

which imply (7). ■