H_∞ Composite Control for Singularly Perturbed Nonlinear Systems via Successive Galerkin Approximation

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Abstract: This paper presents a new algorithm for the closed-loop H_∞ composite control of singularly perturbed nonlinear systems with a exogenous disturbance, using the successive Galerkin approximation(SGA). The singularly perturbed nonlinear system is decomposed into two subsystems of a slow-time scale and a fast-time scale via singular perturbation theory, and two H_∞ control laws are obtained to each subsystem by using the SGA method. The composite control law that consists of two H_∞ control laws of each subsystem is designed. One of the purposes of this paper is to design the closed-loop H_∞ composite control law for the singularly perturbed nonlinear systems via the SGA method. The other is to reduce the computational complexity when the SGA method is applied to the high order systems.

Keywords: H_∞ control, Composite control, Nonlinear system, Singular perturbation, Successive Galerkin Approximation

1. Introduction

Many real physical systems are described by singularly perturbed nonlinear systems [1], [2], [3]. Singularly perturbed systems include two or multi time scales and have been studied by many researchers [2], [3], [4]. In the class of optimal control [5], design of the control law for the singularly perturbed systems has ill-defined numerical problems [2], [3], [4]. To avoid these problems, the full order system is decomposed into reduced slow and fast subsystems, and then optimal control laws are designed for each subsystem. Thus, the near-optimal composite control law consists of two optimal sub-control laws [2], [3], [4]. In addition, recently, robust control is issued and developed by many researchers for linear systems [6], [7] [8], [9]. But in the class of nonlinear systems, conditions for the solvability of the robust H_∞ design problem are hardness and the solution of Hamilton-Jacobi-Bellman(HJB) can be hardly found [10], [11], [12], and thus we will find the approximated solutions using successive Galerkin approximation(SGA) [13], [14].

However, the SGA method has the difficulty that the complexity of computations increases according to order of system. Therefore the full order system is decomposed into the reduced order subsystems via singular perturbation theory and then two robust H_∞ sub-control laws are designed for the corresponding slow and fast nonlinear systems using the SGA method, respectively. The obtained closed-loop H_∞ composite control law is represented by a linear combination of the slow and fast variables.

The purpose of this paper is to design the closed-loop H_∞ composite control laws for singularly perturbed nonlinear systems using the SGA method. In order to obtain the closed-loop H_∞ control law using the SGA method, one must compute $n-$dimensional integrals, and the number of computations increases according to n. Singularly perturbed systems can be decomposed into two subsystems, and we can obtain two sub-control laws for each subsystem through SGA method. Therefore, n_1- and n_2-dimensional integrals are computed and the number of computations are decreased, where $n = n_1 + n_2$. Thus, the near-optimal H_∞ composite control law consists of two optimal H_∞ sub-control laws.

The contents of this paper are as follows. In section 2, singularly perturbed nonlinear systems with respect to performance criteria are studied. We define Generalized-Hamilton-Jacobi-Bellman(GHJB) equations for each subsystems. The solutions of GHJB equations are obtained using the SGA method and the composite H_∞ control law is designed, and we present the new algorithm for H_∞ composite control of singularly perturbed nonlinear systems using the SGA method. In section 3, the proposed algorithm for the singularly perturbed nonlinear systems is applied to a numerical example. Finally, section 4 gives our conclusion.

2. Main Results

The solution of Hamilton-Jacobi-Bellman equations for nonlinear systems can be hardly found, thus we will find the approximated solutions using successive Galerkin approximation. Singularly perturbed nonlinear system is decomposed into two subsystems in the spirit of the general theory of singular perturbation. The H_∞ sub-control laws are designed for each subsystem using the SGA method, and the closed-loop H_∞ composite control law consists of two optimal control laws for each subsystem.

2.1. H_∞ Composite control for singularly perturbed nonlinear systems

The infinite-time H_∞ control problem considers a class of singularly perturbed nonlinear systems described by the following differential equations:

$$\dot{\alpha} = f_1(\alpha) + F_1(\alpha)\alpha + g_1(\alpha)u + h_1(\alpha)\omega \quad (1)$$

$$\epsilon\dot{\beta} = f_1(\alpha) + F_1(\alpha)\beta + g_2(\alpha)u + h_2(\beta)\omega \quad (2)$$

$$z = \begin{bmatrix} l(\alpha) \\ L(\alpha)\beta \\ Du \end{bmatrix} \quad (3)$$

$$\alpha(t_0) = \alpha^0, \qquad \beta(t_0) = \beta^0$$

with respect to the performance criterion:

$$J = \int_0^\infty \left(z^T z - \gamma^2 \omega^T \omega \right) dt \quad (4)$$

where $\alpha \in R^{n_1}$, and $\beta \in R^{n_2}$ are states, $u \in R^m$ is a control input, $\omega \in R^p$ is a exogenous disturbance, and ϵ is a small positive parameter. We assume that $f_1 \in R^{n_1}$, $f_2 \in R^{n_2}$, $F_1 \in R^{n_1 \times n_2}$, $F_2 \in R^{n_2 \times n_2}$, $g_1 \in R^{n_1 \times m}$, $g_2 \in R^{n_2 \times m}$, $h_1 \in R^{n_1 \times p}$, $h_2 \in R^{n_2 \times p}$ are Lipschitz continuous on a compact set $\Omega \supset B(0)$, where B is a ball around the states $[\alpha^T \; \beta^T]^T$. We also assume that $f_1(t_0) = 0$ and $f_2(t_0) = 0$. In addition, for simplification of development we assume as follows:

$$h_2(\alpha) = 0$$

The performance criterion (4) can be written in the equivalent form:

$$J = \int_0^\infty \left(l^T l + \beta^T L^T L \beta + u^T D^T D u - \gamma^2 \omega^T \omega \right) dt \quad (5)$$

In the following, we solve slow and fast optimal control problems and combine their solutions to form a composite control.

$$u_c = u_s^* + u_f^* \quad (6)$$

The near-optimality of the composite control law is stated in the following lemma.

Lemma 1

$$\begin{aligned} u^*(t) &= u_c(t) + O(\epsilon), \quad t \geq t_0 \\ \alpha(t) &= \alpha_s(t) + O(\epsilon), \quad t \geq t_0 \\ \beta(t) &= \alpha_s(t) + \alpha_f(t) + O(\epsilon), \quad t \geq t_0 \end{aligned}$$

The proof of this lemma can be drawn from (Chow and Kokotovic, 1976, [4]]; Kokotovic et al., 1986, [3]).

Let us assume that the open-loop system (1-2) is a standard singularly perturbed system for every $u \in B(u) \subset R^m$, that is, the equation

$$\beta_s = -F_2^{-1}(\alpha_s)\left\{f_2(\alpha_s) + g_2(\alpha_s) u_s\right\} \quad (7)$$

has a unique solution.

The slow time scale problem of order n_1 is defined by eliminating β_f and u_f from (1-3) and (5) using (7). Then the resulting slow time scale problem becomes optimal control of the slow subsystem

$$\dot{\alpha}_s = f_0(\alpha_s) + g_s(\alpha_s) u_s, \quad \alpha_s(t_0) = \alpha^0 \quad (8)$$

with respect to the performance criterion

$$J_s = \int_0^\infty \left\{ l_0(\alpha_s) + 2 L_s(\alpha_s) u_s + u_s^T D_s(\alpha_s) u_s \right\} dt \quad (9)$$

where

$$\begin{aligned} f_0 &= f_1 - F_1 F_2^{-1} f_2 \\ g_s &= g_1 - F_1 F_2^{-1} g_2 \\ l_0 &= l^T l + f_2^T F_2^{-T} L^T L F_2^{-1} f_2 \\ L_s &= f_2^T F_2^{-T} L^T L F_2^{-1} g_2 \, . \\ D_s &= D^T D + g_2^T F_2^{-T} L^L F_2^{-1} g_2 \end{aligned}$$

From robust H_∞ control theory [6], [7], it is well known that if $J_s^*(\alpha_s)$ is a unique positive-definite solution of the Hamilton-Jacobi-Bellman equation

$$0 = l_s + \frac{\partial J^*}{\partial \alpha_s}^T f_s - \frac{1}{4} \frac{\partial J_s^*}{\partial \alpha_s}^T \left(g_s D_s^{-1} g_s^T - \gamma^{-2} h_1 h_1^T \right) \frac{\partial J_s^*}{\partial \alpha_s} \quad (10)$$

with the boundary condition

$$J_s^*(0) = 0 \quad (11)$$

then the H_∞ control of the slow time scale problem is given by

$$u_s^* = -D_s^{-1} \left(L_s^T + \frac{1}{2} g_s^T \frac{\partial J_s^*}{\partial \alpha_s} \right) \quad (12)$$

and the exogenous disturbance of the worst case is given by

$$\omega^* = \frac{\gamma^{-2}}{2} h_1^T \frac{\partial J_s^*}{\partial \alpha_s} \quad (13)$$

where

$$\begin{aligned} f_s &= f_0 - g_s D_s^{-1} L_s^T \\ l_s &= l_0 - L_s D_s^{-1} L_s^T \end{aligned}$$

The fast time scale problem of order n_2 is defined by freezing the slow variable α_s and shifting the equilibrium of the fast subsystem to the origin.

$$\epsilon \dot{\beta}_f = F_2(\alpha_s) \beta_f + g_2(\alpha_s) u_f \quad (14)$$

$$\beta_f(t_0) = \beta^0 + F_2^{-1}(\alpha^0)\{f_2(\alpha^0) + g_2(\alpha^0) u_s(t_0)\}$$

where $\beta_f = \beta - \beta_s$. The performance criterion of the fast time scale problem is given by

$$J_f = \int_0^\infty \left\{ \beta_f^T L^T(\alpha_s) L(\alpha_s) \beta_f + u_f^T D^T D u_f \right\} dt \quad (15)$$

where $\alpha_s \in B$ is fixed parameter.

If $J_f^*(\beta_f)$ is a unique positive-definite solution of the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} 0 &= \beta_f^T L^T L \beta_f + \frac{\partial J_f^*}{\partial \beta_f}^T F_2 \beta_f \\ &\quad - \frac{1}{4} \frac{\partial J_f^*}{\partial \beta_f}^T g_2 (D^T D)^{-1} g_2^T \frac{\partial J_f^*}{\partial \beta_f} \end{aligned} \quad (16)$$

with the boundary condition

$$J_f^*(0) = 0 \quad (17)$$

then the H_∞ control of the fast time scale problem is given by

$$u_f^* = -\frac{1}{2} (D^T D)^{-1} g_2^T \frac{\partial J_f^*}{\partial \beta_f} \, . \quad (18)$$

A realizable composite control requires that the system states α_s and β_f be expressed in terms of the actual system states α and β. This can be achieved by replacing α_s by α and β_f by $\beta - \beta_s$ so that

$$\begin{aligned} u_c &= -D_s^{-1} (L_s^T + \frac{1}{2} g_s^T \frac{\partial J_s^*}{\partial \alpha_s}) - \frac{1}{2} (D^T D)^{-1} g_2^T \frac{\partial J_f^*}{\partial \beta_f} \\ &= -D_s^{-1} (L_s^T + \frac{1}{2} g_s^T G_s \alpha) - \frac{1}{2} (D^T D)^{-1} g_2^T G_f \Big[\beta \\ &\quad + F_2^{-1} f_2 - F_2^{-1} g_2 D_s^{-1} (L_s^T + \frac{1}{2} g_s^T G_s \alpha) \Big] \end{aligned} \quad (19)$$

where $\partial J_s^*/\partial\alpha_s = G_s\alpha_s$ and $\partial J_f^*/\partial\beta_f = G_f\beta_f$.

2.2. GHJB equations for singularly perturbed nonlinear systems

In order to obtain the H_∞ composite control law u_c, we need to find the solutions, $\partial J_s^*/\partial\alpha_s$ and $\partial J_f^*/\partial\beta_f$, using successive Galerkin approximation.

Assumption
Ω is a compact set of R^n, and all states are bounded on Ω.

Under Assumption and by the help of [13], [14], [15], [16], we can define the Generalized-Hamilton-Jacobi-Bellman equation for singular perturbed nonlinear systems which is defined in the following.

Definition
If initial control laws, $\tilde{u}_s^{(0)} : R^m \times \Omega_s \longrightarrow R^m$ and $u_f^{(0)} : R^m \times \Omega_f \longrightarrow R^m$, are admissible and functions, $J_s^{(i)} : R^{n_1} \times \Omega_s \longrightarrow R^{n_1}$ and $J_f^{(i)} : R^{n_2} \times \Omega_f \longrightarrow R^{n_2}$, satisfy the following Generalized-Hamilton-Jacobi-Bellman equations, written by $GHJB(J_s^{(i)}, \tilde{u}_s^{(i)}) = 0$, namely

$$\begin{aligned} 0 &= l_s + \frac{1}{4}\frac{\partial J_s^{(i-1)}}{\partial\alpha_s}^T \left(g_s D_s^{-1} g_s^T - \gamma^{-2} h_1 h_1^T\right) \frac{\partial J_s^{(i-1)}}{\partial\alpha_s} \\ &\quad + \frac{\partial J_s^{(i)}}{\partial\alpha_s}^T f_s - \frac{1}{2}\frac{\partial J_s^{(i)}}{\partial\alpha_s}^T \left(g_s D_s^{-1} g_s^T - \gamma^{-2} h_1 h_1^T\right) \frac{\partial J_s^{(i-1)}}{\partial\alpha_s} \end{aligned} \quad (20)$$

with boundary condition

$$J_s^i(0) = 0 \quad (21)$$

then ith slow control law is

$$\tilde{u}_s^{(i)} = -\frac{1}{2} D_s^{-1} g_s^T \frac{\partial J_s^{(i-1)}}{\partial\alpha_s} \quad (22)$$

and $GHJB(J_f^{(i)}, u_f^{(i)}) = 0$, namely

$$\begin{aligned} 0 &= \beta_f^T L^T L\beta_f + \frac{1}{4}\frac{\partial J_f^{(i-1)}}{\partial\beta_f}^T g_2(D^T D)^{-1} g_2^T \frac{\partial J_f^{(i-1)}}{\partial\beta_f} \\ &\quad + \frac{\partial J_f^{(i)}}{\partial\beta_f}^T F_2\beta_f - \frac{1}{2}\frac{\partial J_f^{(i)}}{\partial\beta_f}^T g_2(D^T D)^{-1} g_s^T \frac{\partial J_f^{(i-1)}}{\partial\beta_f} \end{aligned} \quad (23)$$

with boundary condition

$$J_f^i(0) = 0 \quad (24)$$

then $i$the fast control law is

$$u_f^{(i)} = -\frac{1}{2}(D^T D)^{-1} g_2^T \frac{\partial J_f^{(i-1)}}{\partial\beta_f} \quad (25)$$

where i is iteration number.

2.3. Galerkin projections of the GHJB equations

In this section, we use Galerkin's projection method to derive approximate solutions to the GHJB equations on the compact set, Ω. We find an approximate solution $J_N^{(i)}$ to the equation $GHJB(J^{(i)}, u^{(i)}) = 0$ by letting

$$J_N^{(i)}(x) = \sum_{j=1}^{N} c_j^{(i)}\phi_j(x) \quad (26)$$

where the coefficients c_j are constant in the infinite-time case. Substituting this expression into the GHJB equation results in an approximation error

$$error = GHJB\left(\sum_{j=1}^{N} c_j^{(i)}\phi_j, u^{(i)}\right). \quad (27)$$

The coefficients c_j are determined by setting the projection of the error, (27) on the finite basis, $\{\phi_j\}_1^N$, to zero for all $x \in \Omega$,

$$< GHJB\left(\sum_{j=1}^{N} c_j^{(i)}\phi_j, u^{(i)}\right), \phi_n >_\Omega = 0, \quad n = 1, \cdots, N. \quad (28)$$

Then, (28) becomes N equations with N unknown constants.

To represent (28) by the matrix equations, we define

$$\Phi_N(x) \equiv (\phi_1(x), \cdots, \phi_N(x))^T \quad (29)$$

and let $\nabla\Phi_N$ be the Jacobian Φ_N. If $\eta : R^N \longrightarrow R^N$ is a vectoer valued function, then we define the notation

$$< \eta, \Phi_N >_\Omega \equiv \begin{bmatrix} < \eta_1, \phi_1 >_\Omega & \cdots & < \eta_N, \phi_1 >_\Omega \\ \vdots & \ddots & \vdots \\ < \eta_1, \phi_N >_\Omega & \cdots & < \eta_N, \phi_N >_\Omega \end{bmatrix}$$

where the inner product is defined as

$$< f, g >_\Omega \equiv \int_\Omega f(x)g(x)dx \quad (30)$$

and

$$J_N \equiv \mathbf{c}_N^T \Phi_N \quad (31)$$

with

$$\mathbf{c}_N \equiv (c_1, c_2, \cdots, c_N)^T. \quad (32)$$

Given an initial control $\tilde{u}_s^{(0)}$, we compute an approximation to its cost $J_{sN}^{(0)} = \mathbf{c}_{sN}^{T(0)}\Phi_{sN}$ where $\mathbf{c}_{sN}^{(0)}$ is the solution of Galerkin approximation of GHJB equation (20), i.e.

$$A_s \mathbf{c}_{sN}^{(0)} + b_s = 0 \quad (33)$$

where

$$A_s = <\nabla\Phi_{sN} f_s, \Phi_{sN}>_{\Omega_s} + <\nabla\Phi_{sN}(g_s\tilde{u}_s^{(0)} + h\omega^{(0)}), \Phi_{sN}>_{\Omega_s}$$
$$b_s = < l_s, \Phi_{sN}>_{\Omega_s} + <\tilde{u}_s^{T(0)} D_s \tilde{u}_s^{(0)} - \gamma^2\omega^{(0)T}\omega^{(0)}, \Phi_{sN}>_{\Omega_s}$$

We can compute the updated control law that is based on the approximated solution, $J_{sN}^{(i-1)}$.

$$\tilde{u}_s^{(i)} = -\frac{1}{2} g_s^T \frac{\partial J_s^{(i-1)}}{\partial\alpha_s} = -\frac{1}{2} g_s^T \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)} \quad (34)$$

$$\omega^{(i)} = \frac{\gamma^{-2}}{2} h_1^T \frac{\partial J_s^{(i-1)}}{\partial\alpha_s} = \frac{\gamma^{-2}}{2} h_1^T \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)} \quad (35)$$

Then we can obtain the approximation

$$J_{sN}^{(i)} = \mathbf{c}_{sN}^{T(i)}\Phi_{sN} \quad (36)$$

where $\mathbf{c}_{sN}^{(i)}$ is the solution of

$$A_s\mathbf{c}_{sN}^{(i)} + b_s = 0 \tag{37}$$

where

$$\begin{aligned} A_s &= <\nabla\Phi_{sN} f_s, \Phi_{sN}>_{\Omega_s} - \frac{1}{2} <\nabla\Phi_{sN}(g_s D_s^{-1} g_s^T \\ &\quad -\gamma^{-2} h_1 h_1^T)\nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN}>_{\Omega_s} \\ b_s &= <l_s, \Phi_{sN}>_{\Omega_s} + \frac{1}{4} < \mathbf{c}_{sN}^{T(i-1)} \nabla\Phi_{sN}(g_s D_s^{-1} g_s^T \\ &\quad -\gamma^{-2} h_1 h_1^T)\nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN}>_{\Omega_s} \end{aligned}$$

and i is iteration number.

Similarly, given an initial control $u_f^{(0)}$, we can compute an approximation to its cost $J_{fN}^{(0)} = \mathbf{c}_{fN}^{T(0)}\Phi_{fN}$ where $\mathbf{c}_{fN}^{(0)}$ is the solution of Galerkin approximation of GHJB equation for the fast-time case.

The following lemma states the existence of unique solutions, $\mathbf{c}_{sN}^{(i)}$ and $\mathbf{c}_{fN}^{(i)}$ of Galerkin approximation.

Lemma 2
Suppose that $\{\phi_{sj}\}_1^N$ and $\{\phi_{fj}\}_1^N$ are linearly independent respectively, then A_s and A_f are invertible. Furthermore, existence of the unique solutions is guaranteed.
The proof of this lemma can be drawn from (Randal W. Beard, 1995, [13]).

2.4. The new algorithm of $H\infty$ composite control for singularly perturbed nonlinear systems

The following algorithm shows that the $H\infty$ composite control can be designed by two closed-loop control laws of fast- and slow-subsystem using the SGA method for singularly perturbed nonlinear systems.

Algorithm

Initial Step
Compute

$$A_s = <\nabla\Phi_{sN} f_s, \Phi_{sN}>_{\Omega_s} + <\nabla\Phi_{sN}(g_s\widetilde{u}_s^{(0)} + h\widetilde{\omega}^{(0)}), \Phi_{sN}>_{\Omega_s}$$
$$b_s = <l_s, \Phi_{sN}>_{\Omega_s} + <\widetilde{u}_s^{T(0)} D_s \widetilde{u}_s^{(0)} - \gamma^2 \widetilde{\omega}^{(0)T}\widetilde{\omega}^{(0)}, \Phi_{sN}>_{\Omega_s}$$

and

$$\begin{aligned} A_f &= <\nabla\Phi_{fN} F_2\beta_f, \Phi_{fN}>_{\Omega_f} + <\nabla\Phi_{fN} g_2 u_f^{(0)}, \Phi_{fN}>_{\Omega_f} \\ b_f &= <\beta_f^T L^T L\beta_f, \Phi_{fN}>_{\Omega_f} + <u_f^{T(0)} D^T D u_f^{(0)}, \Phi_{fN}>_{\Omega_f} \end{aligned}$$

Find $\mathbf{c}_{sN}^{(0)}$ and $\mathbf{c}_{fN}^{(0)}$ satisfying the following linear equations:

$$A_s\mathbf{c}_{sN}^{(0)} + b_s = 0, \quad A_f\mathbf{c}_{fN}^{(0)} + b_f = 0$$

Set $i = 1$.

Iterative Step
Improved controllers are given by

$$\begin{aligned} \widetilde{u}_{sN}^{(i)} &= -\frac{1}{2} D_s^{-1} g_s^T \nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)} \\ u_{fN}^{(i)} &= -\frac{1}{2}(D^T D)^{-1} g_2^T \nabla\Phi_{fN}^T \mathbf{c}_{fN}^{(i-1)} \end{aligned}$$

Compute

$$\begin{aligned} A_s &= <\nabla\Phi_{sN} f_s, \Phi_{sN}>_{\Omega_s} - \frac{1}{2} <\nabla\Phi_{sN}(g_s D_s^{-1} g_s^T \\ &\quad -\gamma^{-2} h_1 h_1^T)\nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN}>_{\Omega_s} \\ b_s &= <l_s, \Phi_{sN}>_{\Omega_s} + \frac{1}{4} < \mathbf{c}_{sN}^{T(i-1)} \nabla\Phi_{sN}(g_s D_s^{-1} g_s^T \\ &\quad -\gamma^{-2} h_1 h_1^T)\nabla\Phi_{sN}^T \mathbf{c}_{sN}^{(i-1)}, \Phi_{sN}>_{\Omega_s} \end{aligned}$$

and

$$\begin{aligned} A_f &= <\nabla\Phi_{fN} F_s\beta_f, \Phi_{fN}>_{\Omega_f} \\ &\quad -\frac{1}{2} <\nabla\Phi_{fN} g_2 R^{-1} g_2^T \nabla\Phi_{fN}^T \mathbf{c}_{fN}^{(i-1)}, \Phi_{fN}>_{\Omega_f} \\ b_f &= <\beta_f^T Q\beta_f, \Phi_{fN}>_{\Omega_f} \\ &\quad +\frac{1}{4} <\mathbf{c}_{fN}^{T(i-1)} \nabla\Phi_{fN} g_2 R^{-1} g_2^T \nabla\Phi_{fN}^T \mathbf{c}_{fN}^{(i-1)}, \Phi_{fN}>_{\Omega_f} \end{aligned}$$

Find $\mathbf{c}_{sN}^{(i)}$ and $\mathbf{c}_{fN}^{(i)}$ satisfying the following linear equations:

$$\begin{aligned} A_s\mathbf{c}_{sN}^{(i)} + b_s = 0 \\ A_f\mathbf{c}_{fN}^{(i)} + b_f = 0 \end{aligned}$$

Set $i = i + 1$.

Final Step
The H_∞ composite control law is

$$\begin{aligned} u_c &= -D_s^{-1}(L_s^T + \frac{1}{2}\, g_s^T \frac{\partial J_s^*}{\partial\alpha_s}) - \frac{1}{2}(D^T D)^{-1} g_2^T \frac{\partial J_f^*}{\partial\beta_f} \\ &= -D_s^{-1}(L_s^T + \frac{1}{2}\, g_s^T G_s\alpha) \\ &\quad -\frac{1}{2}(D^T D)^{-1} g_2^T G_f \Big[\beta + F_2^{-1} f_2 \\ &\quad -F_2^{-1} g_2 D_s^{-1}\big(L_s^T + \frac{1}{2}\, g_s^T G_s\alpha\big)\Big] \end{aligned}$$

where $\nabla\Phi_{sN}^T \mathbf{c}_{sN} = G_s\alpha_s$ and $\nabla\Phi_{fN}^T \mathbf{c}_{fN} = G_f\beta_f$.

The following lemma shows that the solution of Galerkin approximation converges to solution of Generalized-Hamilton-Jacobi-Bellman equation.

Lemma 3
For any small positive constant α, we can choose N and i sufficiently large to satisfy that

$$\|J^* - J_N^{(i)}\| < \alpha \tag{38}$$

The proof of this lemma can be easily drawn from (Randal W. Beard, [13]).

3. A Numerical Example

In this section, we apply the proposed algorithm to a numerical example. Consider the fifth-order numerical example which is the standard singularly perturbed nonlinear system (1-3). The states variables are $\alpha = [x_1^T\ x_2^T\ x_3^T]^T$ and $\beta = [x_4^T\ x_5^T]^T$, and the control variable is $u = [u_1^T\ u_2^T]^T$.

$$\begin{aligned} \dot{\alpha} &= f_1(\alpha) + F_1(\alpha)\beta + g_1(\alpha)u + h_1(\beta), & \alpha(t_0) = \alpha^0 \\ \epsilon\dot{\beta} &= f_2(\alpha) + F_2(\alpha)\beta + g_2(\alpha)u, & \beta(t_0) = \beta^0 \end{aligned}$$

Where the problem matrices have the following values.

$$f_1(\alpha) = \begin{bmatrix} -0.04611x_1 \\ -2.146x_1 - x_1x_3 \\ x_1x_2 - 2.146x_3 \end{bmatrix},$$

$$F_1(\alpha) = \begin{bmatrix} -16.6x_3 & 16.6x_2 \\ 0.146 & 0 \\ 0 & 0.146 \end{bmatrix},$$

$$f_2(\alpha) = \begin{bmatrix} 0.146x_2 + 0.068x_1x_3 \\ -0.068x_1x_2 + 0.146x_3 \end{bmatrix},$$

$$F_2(\alpha) = \begin{bmatrix} -0.00225 & 0 \\ 0 & -0.00225 \end{bmatrix},$$

$$g_1(\alpha) = 0, \quad g_2(\alpha) = 0.0399I_2$$

$$h_1(\alpha) = [1\ 0\ 0]^T, \quad \epsilon = 0.000262$$

In this paper, we assume that the exogenous disturbance, $\omega = 130sin(148pit)$. The simulation results are presented with initial states, $x_0 = [10\ -0.07\ 0.04\ 15\ 47]^T$, in the figures 1-6 where the dashed lines (– –) are the trajectories that obtained from full-order SGA method and the sold lines (——) are the trajectories that obtained from the proposed algorithm. The figure 6 shows that the performance criterion trajectory of the proposed algorithm is better than that of the full-order SGA method, because errors of the full-order SGA method are bigger than those of the proposed algorithm. In the full-order SGA method, ten-dimensional basis are used and five-dimensional integrals of $10\times(1+10+100) = 1110$ times are performed. But, in the proposed algorithm, we can use only six-dimensional basis and compute three-dimensional integrals of $6 \times (1 + 6 + 36) = 248$ times for slow-time scale subsystem, and compute two-dimensional integrals of $3\times(1+3+9) = 39$ times based on three-dimensional basis for fast-time scale subsystem in parallel. Therefore, the computational complexity is greatly reduced.

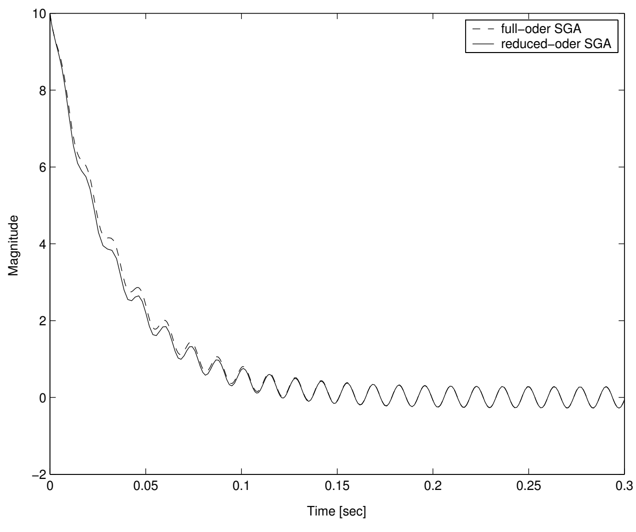

Fig. 1. Trajectories of α

4. Conclusion

In this paper, we have presented the closed-loop H_∞ composite control scheme of singularly perturbed nonlinear systems

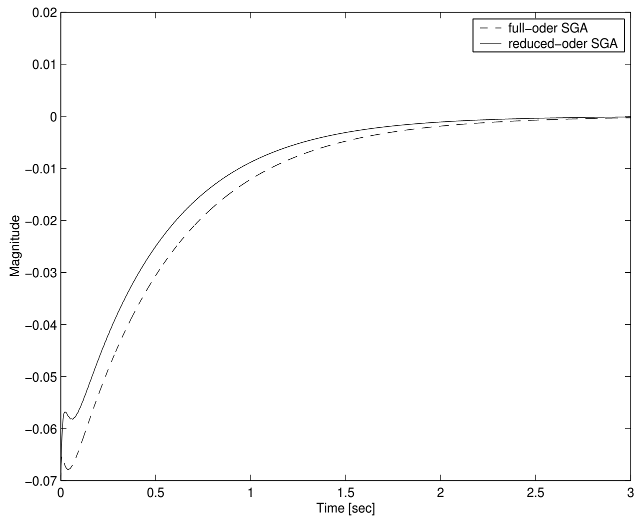

Fig. 2. Trajectories of β

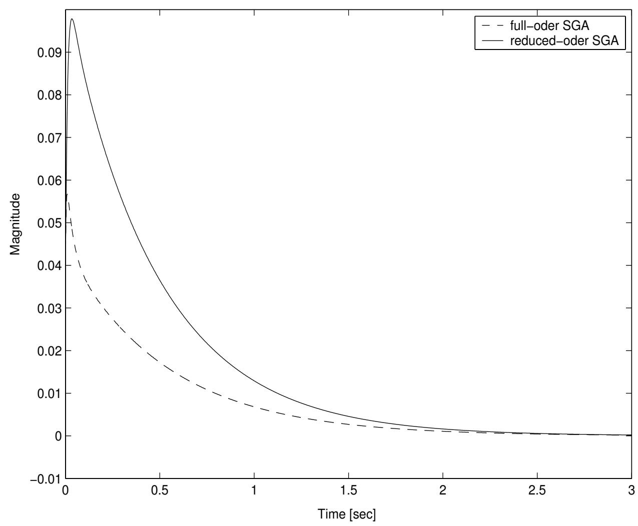

Fig. 3. Trajectories of x_3

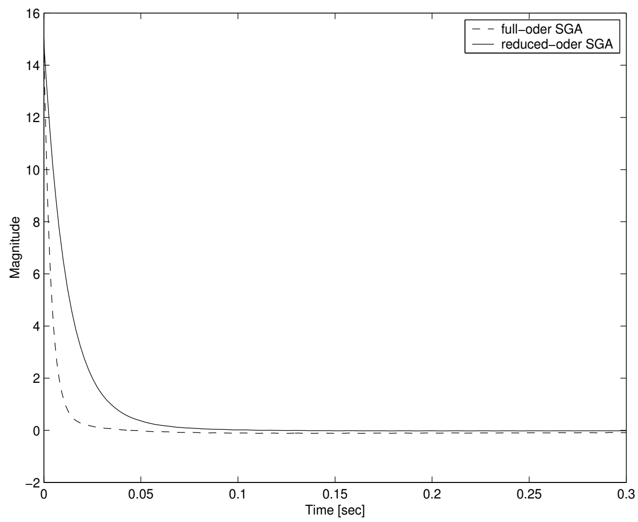

Fig. 4. Trajectories of x_4

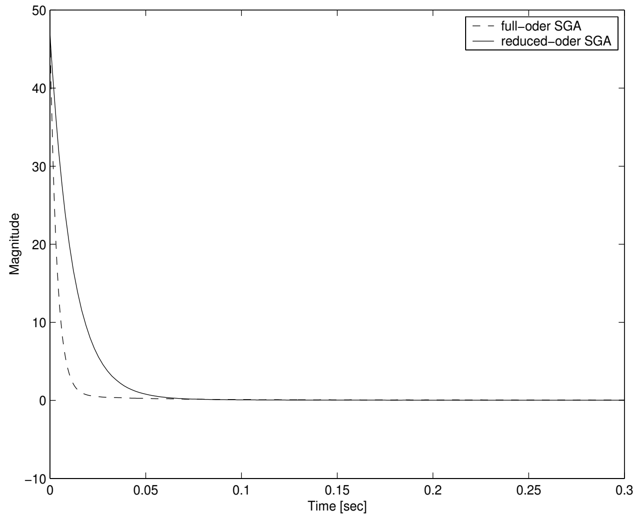

Fig. 5. Trajectories of x_5

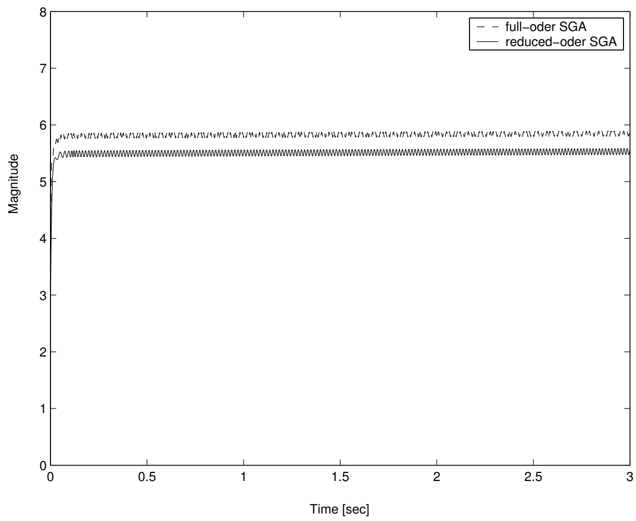

Fig. 6. Trajectories of performance index

using the successive Galerkin approximation(SGA). The difficulty of the SGA method is a computational complexity, but in the proposed algorithm, $n-$dimensional integrals are reduced to n_1- and n_2-dimensional integrals and the computational complexity according to n states is decreased to n_1 and n_2 states. The presented simulation results for singularly perturbed nonlinear systems show that the performance trajectories of the proposed algorithm is better than those of the full order SGA method. In addition, it should be noted that the proposed algorithm are more effective than the full order SGA method.

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