Constrained Robust Model Predictive Control with Enlarged Stabilizable Region

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Abstract: The dual-mode strategy has been adopted in many constrained MPC methods. The size of stabilizable regions of states of MPC methods depends on the size of underlying feasible and positively invariant set and number of control moves. These results, however, could be conservative because the definition of positive invariance does not allow temporal leave of states from the set. In this paper, a concept of periodic invariance is introduced in which states are allowed to leave a set temporarily but return into the set in finite time steps. The periodic invariance can be defined with respect to sets of different state feedback gains. These facts make it possible for the periodically invariant sets to be considerably larger than ordinary invariant sets. The periodic invariance can be defined for systems with polyhedral model uncertainties. We derive a MPC method based on these periodically invariant sets. Some numerical examples are given to show that the use of periodic invariance yields considerably larger stabilizable sets than the case of using ordinary invariance.

Keywords: periodic invariance, input constraints, model uncertainty, receding horizon control

1. Introduction

The 'dual-mode paradigm' is known to be an effective way to handle physical constraints in actuators.[1]-[4] The basic idea of the dual-mode paradigm is to use feasible control moves to steer the current state into a feasible and invariant set in finite time steps. A constant state feedback control is assumed to be used after the state belongs to the positively invariant set. The feasible and positive invariance of a set is defined with respect to a state feedback gain and it requires that the state feedback control should satisfy the input constrains for all the states in the set and states should remain in the set when the state feedback control is applied. This dual-mode strategy has been adopted in many constrained MPC methods. The size of stabilizable regions of states of MPC methods depends on the size of underlying feasible and positively invariant set and number of control moves. These results, however, could be conservative because the definition of positive invariance does not allow temporal leave of states from the set. Motivated by these considerations, the concept of quasi-invariant sets was introduced in [3], which allows the state to leave the set temporarily. The approach used in [3] is based on polyhedral type terminal sets. In [3], however, no systematic way of obtaining underlying state feedback gain is provided and a heavy computational burden is required.

In this paper, a concept of periodic invariance is introduced in which states are allowed to leave a set temporarily but return into the set in finite time steps. Moreover, the periodic invariance involves the use of more than one state feedback gains and several ellipsoidal sets. These facts make it possible for the periodically invariant sets to be considerably larger than ordinary invariant sets. The periodic invariance can be defined for systems with polyhedral model uncertainties. We derive a MPC method based on these periodically invariant sets. In section 2, the periodic invariance is defined and a MPC method that uses the convex hull of the positively invariant sets as a target was developed in Section 3. A Lyapunov function defined as a sum of quadratic function and it will be shown that this Lyapunov function can be made monotonically decreasing.

2. Periodic Invariance and Feasibility

Consider the following input constrained linear uncertain system:

$$\mathbf{x}(k+1) = \tilde{A}\mathbf{x}(k) + \tilde{B}\mathbf{u}(k),\ |\mathbf{u}(k)| \leq \overline{\mathbf{u}}, \tag{1}$$

where the matrix functions $\tilde{A}$ and $\tilde{B}$ belong to the polyhedral uncertainty class:

$$\Omega = \left\{ (\tilde{A}, \tilde{B}) | (\tilde{A}, \tilde{B}) = \sum_{l=1}^{n_p} \eta_l (A_l, B_l),\ \eta_l \geq 0, \sum_{l=1}^{n_p} \eta_l = 1 \right\}, \tag{2}$$

We will consider a time-varying state feedback control law as:

$$\mathbf{u}(k) = K(k)\mathbf{x}(k), \tag{3}$$

which requires

$$|\mathbf{u}(k)| = |K(k)\mathbf{x}(k)| \leq \overline{\mathbf{u}}. \tag{4}$$

Provided that (4) is satisfied, use of $\mathbf{u}(k) = K(k)\mathbf{x}(k)$ would yield

$$\mathbf{x}(k+1) = \tilde{\Phi}(k)\mathbf{x}(k), \quad \tilde{\Phi}(k) := \tilde{A} + \tilde{B}K(k). \tag{5}$$

Consider the uncertain linear system described by (1) and (2). A set Ω_0 is defined to be feasible and **periodic-invariant** with respect to the time varying feedback control $\mathbf{u}(k) = K(k)\mathbf{x}(k)$ of (3) if there exists a finite positive number $\mathbf{v}$ such that for any initial state $\mathbf{x}(k) \in \Omega_0$, the future states $\mathbf{x}(k+i)$ $(i = 1, \cdots, \mathbf{v})$ of the system (5) satisfy the input constraint (4)(feasible) and $\mathbf{x}(k+\mathbf{v})$ belongs to Ω_0(periodic-invariant).
Consider an ellipsoidal set defined as:

$$\Omega_0 = \{\mathbf{x} | \mathbf{x}' P_0 \mathbf{x} \leq 1\}. \tag{6}$$

The periodic-invariance of Ω_0 would be checked by considering propagation of the states in terms of ellipsoidal sets. Assume that the closed-loop dynamics of (5) makes $\mathbf{x}(k+1) \in \Omega_1$ for any $\mathbf{x}(k) \in \Omega_0$, where

$$\Omega_1 := \{\mathbf{x} | \mathbf{x}' P_1 \mathbf{x} \leq 1\}. \tag{7}$$

It is easy to see that the following relation:

$$P_0 - \Phi_l(k)' P_1 \Phi_l(k) > 0, \quad l = 1, 2, \cdots, n_p \tag{8}$$

guarantees that $\mathbf{x}(k+1) \in \Omega_1$ for any $\mathbf{x}(k) \in \Omega_0$ and $(\tilde{A}, \tilde{B}) \in \Omega$, where $\Phi_l(k) := A_l + B_l K(k)$. Similarly, an ellipsoidal set Ω_2 can be defined for the ellipsoidal set Ω_1. Relations

$$P_1 - \Phi_l(k+1)' P_2 \Phi_l(k+1) > 0, \quad l = 1, 2, \cdots, n_p \tag{9}$$

would guarantee that $\mathbf{x}(k+2) \in \Omega_2$ for any $\mathbf{x}(k+1) \in \Omega_1$ and $(\tilde{A}, \tilde{B}) \in \Omega$.
The above argument can be applied recursively to yield ellipsoidal sets of states:

$$\Omega_j = \{\mathbf{x} | \mathbf{x}' P_j \mathbf{x} \leq 1\}, \tag{10}$$

and relations

$$P_j - \Phi_l(k+j)' P_{j+1} \Phi_l(k+j) > 0, \quad l = 1, 2, \cdots, n_p, \tag{11}$$

for $j = 1, 2, \cdots, \mathbf{v}-1$. The periodic-invariance of Ω_0 requires that $\Omega_{\mathbf{v}}$ should belong back to Ω_0. Thus, relation

$$P_{\mathbf{v}} - P_o > 0 \tag{12}$$

would guarantee the periodic-invariance of Ω_0 with respect to the switching control (3). On the other hand, it should be noted that the above arguments hold true for the system (1) provided that

$$|K(k+j)\mathbf{x}| \leq \overline{\mathbf{u}}, \quad \forall \mathbf{x} \in \Omega_j, \quad j = 0, 1, \cdots, (\mathbf{v}-1). \tag{13}$$

Conditions (8), (9), (11), (12) and (13) can be transformed into LMIs using technique proposed in [6] and used in [4], which technique is well known and the corresponding LMIs are summarized in the following theorem without proof:

Theorem 1: Consider the constrained uncertain system (1-2). An ellipsoidal set:

$$\Omega_0 = \{\mathbf{x} | \mathbf{x}' P_0 \mathbf{x} \leq 1\} \tag{14}$$

is feasible and periodic-invariant with respect to the time-varying control (3) provided that there exist matrices $Q_j := P_j^{-1} (> 0)(j = 0, 1, 2, \cdots, \mathbf{v})$, and Y_j, $X_j (j = 0, 1, 2, \cdots, \mathbf{v}-1)$ such that $Y_j = K(k+j) \cdot Q_j$,

$$\begin{bmatrix} Q_{j-1} & (A_l Q_{j-1} + B_l Y_{j-1})^T \\ (A_l Q_{j-1} + B_l Y_{j-1}) & Q_j \end{bmatrix} > 0, \tag{15}$$

for $l = 1, 2, \cdots, n_p$ and $j = 1, 2, \cdots, \mathbf{v}$,

$$\begin{bmatrix} Q_{\mathbf{v}} & Q_{\mathbf{v}} \\ Q_{\mathbf{v}} & Q_0 \end{bmatrix} > 0, \tag{16}$$

and

$$\begin{bmatrix} X_{j,m,p} & Y_{j,m,p} \\ Y_{j,m,p}^T & Q_{j,m} \end{bmatrix} > 0, \; X_{j,m,p} \leq \overline{\mathbf{u}}, \tag{17}$$

for $j = 0, 1, 2, \cdots, \mathbf{v}-1$.

The relaxation of the definition of invariance through the introduction of periodic invariance allows the state to leave Ω_0 for a period steering it back to Ω_0 after $\mathbf{v}$ moves. This in turn allows for the enlargement of the volume of Ω_0 which can be achieved through the convex optimisation:

Algorithm 1

$$\min_{Q_j, X_j, Y_j} \; -log(det(Q_0)) \quad subject\ to\ (15-17) \tag{18}$$

3. Receding Horizon Control Based on the Periodic Invariance

The optimisation of Algorithm 1 was aimed exclusively at the minimization of $-log(det(P_0^{-1}))$ with the view to enlarging the volume of Ω_0. The sizes of accompanied ellipsoids Ω_j, $j = 1, 2, \cdots, \mathbf{v}-1$ are expected to be large also. Consider the convex hull Ξ of the ellipsoids Ω_j, $j = 0, 1, 2, \cdots, \mathbf{v}-1$. It is clear that Ξ is larger than the union of the ellipsoids Ω_j, $j = 0, 1, 2, \cdots, \mathbf{v}-1$. Furthermore, Ξ is invariant in the sense that there exists a feasible control input $\mathbf{u}(k)$ which makes the current state $\mathbf{x}(k) \in \Xi$ remain in Ξ as per the following Lemma.

LEMMA 1: Consider the uncertain system (1-2) and ellipsoidal sets Ω_j, $j = 0, 1, \cdots, \mathbf{v}-1$ defined as (10-13). Denote the convex hull of Ω_j, $j = 0, 1, \cdots, \mathbf{v}-1$ as Ξ. If a state $\mathbf{x}(k)$ belongs to Ξ, then there exist a feasible control input $\mathbf{u}(k)$ which guarantees that $\mathbf{x}(k+1) \in \Xi$.

Proof : The state $\mathbf{x}(k) \in \Xi$ can be represented as:

$$\mathbf{x}(k) = \sum_{j=0}^{\mathbf{v}} \lambda_j \mathbf{x}_j(k), \quad \sum_{j=0}^{\mathbf{v}} = 1, \; \lambda_j \geq 0 \tag{19}$$

where $\mathbf{x}_j(k) \in \Omega_j$. Consider the control input $\mathbf{u}(k) = \sum_{j=0}^{\mathbf{v}-1} K_j \lambda_j \mathbf{x}_j(k)$, then $\mathbf{x}(k+1)$ can be represented as:

$$\begin{aligned} \mathbf{x}(k+1) &= \tilde{A}\mathbf{x}(k) + \tilde{B}\mathbf{u}(k) \qquad (20) \\ &= \sum_{j=0}^{\mathbf{v}-1} \lambda_j (\tilde{A} + \tilde{B} K_j) \mathbf{x}_j(k) = \sum_{j=0}^{\mathbf{v}-1} \lambda_j \mathbf{x}_j(k+1). \end{aligned}$$

From the definition of Ω_j, $j = 0, 1, \cdots, \mathbf{v}-1$, $\mathbf{x}_j(k+1) = (\tilde{A} + \tilde{B}) \mathbf{x}_j(k) \in \Omega_{j+1}$. Thus, it is easy to see that $\mathbf{x}(k+1) \in \Xi$ also and we can conclude that there always exist a feasible state feedback law that makes $\mathbf{x}(k)$ remain in Ξ.

Based on the above argument, we would like to propose a MPC strategy using Ξ as a target set. Assume that Ω_0, corresponding ellipsoids Ω_j, $j = 1, 2, \cdots, \mathbf{v}-1$ and their convex hull Ξ were obtained by solving (18). Our control strategy is to steer the current state into Ξ using a feasible control move $\mathbf{u}(k)$. According to the uncertainties (2) reside in the system, $\mathbf{x}(k+1)$ would belong to the polyhedral set of states defined as:

$$\mathcal{F} := \left\{ \mathbf{x} \in R^n | \sum_{l=1}^{n_p} \eta_l (A_l \mathbf{x}(k) + B_l \mathbf{u}(k)) \; \eta_l \geq 0, \; \sum_{l=1}^{n_p} \eta_l = 1 \right\}. \tag{21}$$

It is easy to see that $\mathcal{F} \in \Xi$ is guaranteed if and only if all the vertices of $\mathcal{F}$ *i.e.* $A_l\mathbf{x}(k) + B_l\mathbf{u}(k)$ belongs to Ξ. If $\mathbf{x}_l(k+1) := A_l\mathbf{x}(k)+B_l\mathbf{u}(k) \in \Xi$, then $\mathbf{v}_l$ can be represented as:

$$\mathbf{x}_l(k+1) = \sum_{j=0}^{\mathbf{v}-1} \lambda_{l,j}\mathbf{x}_{l,j}(k+1), \quad \left(\sum_{j=0}^{\mathbf{v}-1} \lambda_{l,j} = 1\right), \tag{22}$$

where $\mathbf{x}_{l,j}(k+1) \in \Omega_j$. If we denote $\lambda_{l,j}\mathbf{x}_{l,j}$ as $\hat{\mathbf{x}}_{l,j}$ then the conditions (22) and $\mathbf{x}_{l,j} \in \Omega_j$ can be rewritten as:

$$A_l\mathbf{x}(k) + B_l\mathbf{u}(k) = \sum_{j=0}^{\mathbf{v}-1} \hat{\mathbf{x}}_{l,j}(k+1), \quad |\mathbf{u}(k)| \leq \overline{\mathbf{u}} \tag{23}$$

and

$$\hat{\mathbf{x}}_{l,j}' \frac{P_j}{\lambda_{l,j}} \hat{\mathbf{x}}_{l,j} \leq \lambda_{l,j}, \tag{24}$$

respectively, for $l = 1, 2, \cdots, n_p$. Thus, the existence of vectors $\mathbf{x}_{l,j}(k+1)$ and scalar values λ_{jl} for $l = 1, 2, \cdots, n_p$ and $j = 0, 1, 2, \cdots, \mathbf{v}-1$ satisfying (23-24) guarantees that $\mathbf{x}(k+1) \in \Xi$.
The control input $\mathbf{u}(k)$ satisfying (23-24) would not be unique. Thus, we need certain criteria to choose a particular $\mathbf{u}(k)$ that is optimal in some sense. Consider the state decomposition (23) and define a quadratic function:

$$V(\mathbf{x}_l(k+1|k)) := \sum_{j=0}^{\mathbf{v}-1} \hat{\mathbf{x}}_{l,j}(k+1)' P_j \hat{\mathbf{x}}_{l,j}(k+1). \tag{25}$$

We would like to use $V(\mathbf{x}_l(k+1|k))$ as our cost index and the control move $\mathbf{u}(k)$ will be chosen to be:

$$\mathbf{u}^*(k) = arg \left\{ \begin{array}{l} min \;\; max \;\; V(\mathbf{x}_l(k+1|k)) \\ \mathbf{u}(k) \;\; l, \hat{\mathbf{x}}_{l,j} \end{array} \right\} \tag{26}$$

$$subject\ to\ (23-24).$$

According to the relations (11-12), $V(\mathbf{x}_l(k+1|k))$ can be made monotonically decreasing, which will be shown later. Note that the relations (23-24) can be rewritten as LMIs in terms of $\mathbf{u}(k)$, $\hat{\mathbf{x}}_{l,j}(k+1)$, and $\lambda_{l,j}$. Now the receding horizon control method based on the above argument can be described as follows:
Algorithm 2
Step 1. (off-line) Obtain matrices $P_j, j = 0, 1, \cdots, \mathbf{v}$ and corresponding ellipsoidal sets $\Omega_j,\ j = 0, 1, \cdots, \mathbf{v}-1$ according to Algorithm 1.
Step 2. (on-line) For a given current state $\mathbf{x}(k)$ compute the optimal control $\mathbf{u}(k)$ as (26), where (23-24) can be rewritten as the following LMIs:

$$diag\left(A_l\mathbf{x}(k) + B_l\mathbf{u}(k) - \sum_{j=0}^{\mathbf{v}-1} \hat{\mathbf{x}}_{l,j}(k+1)\right) = 0 \tag{27}$$

$$\begin{bmatrix} \lambda_{l,j} & \hat{\mathbf{x}}_{l,j}(k+1)' \\ \hat{\mathbf{x}}_{l,j}(k+1) & \lambda_{l,j}Q_j \end{bmatrix} > 0, \tag{28}$$

$$j = 0, 1, \cdots, \mathbf{v}-1, \;\; l = 1, 2, \cdots, n_p$$

$$diag(\overline{\mathbf{u}} - \mathbf{u}(k)) > 0 \tag{29}$$

Closed-loop stability of the Algorithm 2 can be established as per the following theorem:

Theorem 2: Consider the uncertain system (1-2). Assume that matrices $P_j, j = 0, 1, \cdots, \mathbf{v}$ and corresponding ellipsoidal sets $\Omega_j,\ j = 0, 1, \cdots, \mathbf{v}-1$ were obtained as Step 1 of Algorithm 2 and Step 2 was feasible at the initial time step, then Step2 of Algorithm 2 remain feasible and the use of the optimal control $\mathbf{u}(k)$ obtained at each time steps guarantees the asymptotic stability of the closed-loop system.
Proof : Feasibility : Conditions (23-24) guarantees that $\mathbf{x}(k+1) \in \Xi$ for a given $\mathbf{x}(k)$. Once the state is steered into Ξ, (23-24) would have feasible solutions for all the subsequent time steps since Ξ is invariant as Lemma 1.
Stability : Assume that $\mathbf{x}(k+1) = \sum_{j=0}^{\mathbf{v}-1} \lambda_j\mathbf{x}_j(k+1)$ and $\mathbf{x}_j \in \Omega_j$ for $j = 0, 1, \cdots, \mathbf{v}-1$. Consider the control input $\mathbf{u}(k+1) = \sum_{j=0}^{\mathbf{v}-1} K_j\lambda_j\mathbf{x}_j(k+1)$, then $\mathbf{x}(k+2)$ can be represented as:

$$\begin{aligned} \mathbf{x}(k+2) &= \tilde{A}\mathbf{x}(k+1) + \tilde{B}\mathbf{u}(k) \\ &= \sum_{j=0}^{\mathbf{v}-1} (\tilde{A} + \tilde{B}K_j)\hat{\mathbf{x}}_j(k+1). \end{aligned} \tag{30}$$

From relations (11-12), we have:

$$V(\mathbf{x}(k+2)) - V(\mathbf{x}(k+1)) \leq 0, \tag{31}$$

where

$$\begin{aligned} & V(\mathbf{x}(k+2)) := \\ & \sum_{j=0}^{\mathbf{v}-2} \mathbf{x}_j(k+1)'(\tilde{A} + \tilde{B}K_j)P_{j+1}(\tilde{A} + \tilde{B}K_j)\mathbf{x}_j(k+1) \\ & \quad + \hat{\mathbf{x}}_{\mathbf{v}-1}(k+1)'(\tilde{A} + \tilde{B}K_j)'P_0(\tilde{A} + \tilde{B}K_j)\hat{\mathbf{x}}_{\mathbf{v}-1}(k+1) \\ & V(\mathbf{x}(k+1)) := \sum_{j=0}^{\mathbf{v}-1} \mathbf{x}_j(k+1)'P_j\mathbf{x}_j(k+1). \end{aligned}$$

Thus, we can conclude that $V_l(\mathbf{x}(\cdot))$ can be made monotonically decreasing.

4. Numerical Examples

Consider the uncertain system [2][4] with polyhedral set Ω defined by (2) with $\overline{u} = 1$

$$A_1 = \begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.0591 & 0.2641 \\ 1.7971 & 0.8717 \end{bmatrix}$$
$$B = \begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix}. \tag{32}$$

Fig.1 shows stabilizable sets with $n_{inv} = 3$, 5, and 10. This figure shows that by increasing n_{inv}, we can get considerable increase of volume of stabilizable set. The resulting region is bigger than those of earlier works[2][4].

5. Conclusions

A receding horizon control strategy was developed for input constrained linear uncertain systems based on periodically invariant sets. The definition of periodically invariant set allows state to leave the set temporarily. An ellipsoidal set is said to be periodically invariant if there is a series of feedback gains such that the use of these gains guarantees that all the states in the set return into the set in finite time steps. The convex hull of these periodically invariant sets can be shown to be positively invariant in the sense that there exists a feasible input that makes the states remain in the convex hull.

A receding horizon control strategy in which the current state is steered into the convex hull of periodically invariant sets was proposed. A Lyapunov function is defined as a sum of quadratic functions and it was shown this Lyapunov function can be made monotonically decreasing by using a nonlinear control law based on the partitioning of the current state and applying different feedback gains for the partitioned states.

The invariant set used in this paper contains the ellipsoidal invariant sets in the earlier works[4][5] as a special case. It will provide a larger invariant set and larger stabilizable set in turn.

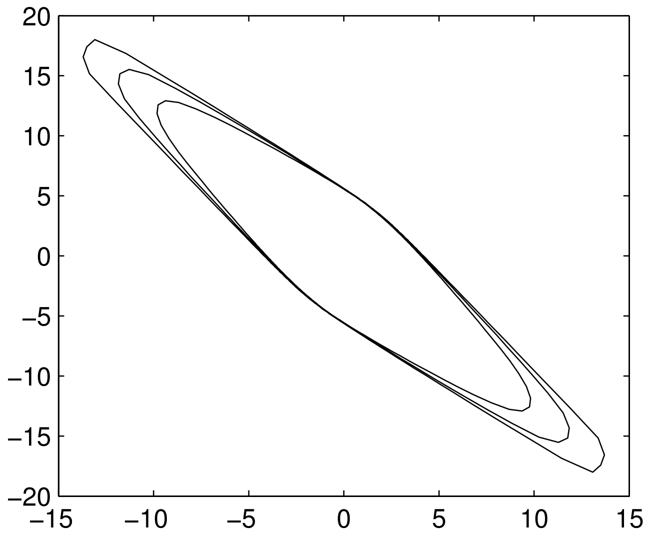

Fig. 1. Stabilizable regions of states with $n_{inv} = 3$(inner line), 5, and 10(outer line).

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