Effect of Element Thickness on the Eigenvalues of Beams

Gun-Myung Lee and Young-Hyo Park *

보 요소의 두께가 고유치에 미치는 영향

이건명 · 박영효

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Abstract

The sensitivities of eigenvalues to the change of element thickness have been calculated for beams in the paper. For a cantilever beam the sensitivities fluctuate more for higher modes. When the thickness of the element near the fixed end increases, the eigenvalues for all modes increase. On the other hand, increasing of the thickness of the element at the tip decreases the eigenvalues for all modes. For a simply supported beam the sensitivities fluctuate more for higher modes, which is the same phenomenon as for a cantilever beam. The sensitivities are always positive for all modes

1. Introduction

Finite Element (FE) analysis is widely used to predict the dynamic responses of mechanical systems and structures subject to dynamic loading. The predicted responses may differ from the experimentally measured ones and there have been active researches on finite element model updating(1) so that the predicted responses based on the model agree with the measured ones. The related researches are surveyed(2) and summarized(3) in references. One of the approaches to model updating is the sensitivity analysis(4). In the approach the sensitivities of the model responses, for example eigenvalues (natural frequencies) and eigenvectors (mode shapes) of the FE model, to changes in the updating parameters are calculated. And the updating parameters of the model are modified according to the sensitivities. Material properties, physical dimensions or joint parameters can be selected as updating parameters(5). Another choice is element correction factors which are multiplied to each element mass and stiffness matrices to modify the FE model(6). This paper investigates some characteristics of the eigenvalue sensitivities to element thickness for beams with various boundary conditions.

† School of Mechanical and Aerospace Engineering, Gyeongsang National University
E-mail : gmlee@gsnu.ac.kr
TEL : (055)751-5313 FAX : (055)757-5622
* Department of Mechanical Engineering, Gyeongsang National University

2. Sensitivity of the eigenvalues

Using the FE analysis(7), the stiffness matrix of the j-th element of a cantilever beam is

$$[K_{ej}] = \frac{Ebh_j^3}{12l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{1}$$

where l, b, and h_j represent the length, width, and thickness of the element, respectively. The mass matrix of the element becomes

$$[M_{ej}] = \frac{\rho b h_j l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{2}$$

Letting the element matrices $[M_{ej}]$ and $[K_{ej}]$ have the same sizes as the whole system matrices, with zeros outside the corresponding positions, and rows and columns deleted for fixed boundary conditions, the system mass and stiffness matrices are expressed by the summation of element matrices as follows.

$$[M]=\sum_{j=1}^{N}[M_{ej}] \quad (3)$$

$$[K]=\sum_{j=1}^{N}[K_{ej}], \quad (4)$$

where N is the number of elements.

It is known that the sensitivity of the eigenvalue (square of the natural frequency) λ_i of mode i to change in the updating parameter θ_j is expressed by Eq. (5)

$$\frac{\partial\lambda_i}{\partial\theta_j}=\phi_i^T\left(\frac{\partial[K]}{\partial\theta_j}-\lambda_i\frac{\partial[M]}{\partial\theta_j}\right)\phi_i, \quad (5)$$

where ϕ_i represents the mass normalized eigenvector of mode I[8]. If we take element thickness h_j as updating parameters, we obtain

$$\frac{\partial[K]}{\partial h_j}=\frac{Ebh_j^2}{4l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix} \quad (6)$$

$$\frac{\partial[M]}{\partial h_j}=\frac{\rho bl}{420}\begin{bmatrix}156 & 22l & 54 & -13l\\ 22l & 4l^2 & 13l & -3l^2\\ 54 & 13l & 156 & -22l\\ -13l & -3l^2 & -22l & 4l^2\end{bmatrix} \quad (7)$$

If we insert Eqs. (6) and (7) into Eq. (5), the sensitivities of eigenvalues are obtained.

3. Results

3.1 Cantilever beam

The element mass and stiffness matrices were formed and the eigenvalues and eigenvectors were calculated for a cantilever beam with length 270 mm, width 35 mm, thickness 1.5 mm, Young's modulus 175×10^9 N/m^2, and density 7850 kg/m^3. The beam is composed of five beam elements with equal length and is shown in Fig. 1. The sensitivities of eigenvalues to the element thickness in Eq. (5) were calculated and are listed in Table 1.

The natural frequencies were calculated for the above cantilever beam. Then the thickness of one of the 5 elements was increased by 10% with the other thickness unchanged and the natural frequencies were calculated for each case. The calculated natural frequencies for the 6 cases are listed in Table 2. Observing the variation of the natural frequencies, it can be found that the variation agrees with the sensitivities in Table 1.

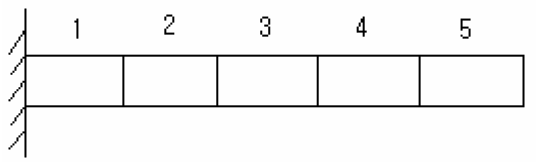

Fig. 1 Cantilever beam composed of five elements

Table 1 Sensitivity of the eigenvalues to the element thickness for a cantilever beam (units : rad^2/s^2)

Element j	1	2	3	4	5
$\frac{\partial\lambda_1}{\partial h_j}$	1.166e7	5.424e6	1.256e6	-1.505e6	-3.878e6
$\frac{\partial\lambda_2}{\partial h_j}$	2.105e8	2.026e7	1.933e8	1.444e8	-5.883e7
$\frac{\partial\lambda_3}{\partial h_j}$	9.676e8	8.743e8	2.948e8	1.772e9	1.121e8
$\frac{\partial\lambda_4}{\partial h_j}$	3.449e9	2.193e9	4.189e9	3.366e9	2.494e9

The above cantilever beam was divided into 20 elements with equal length and the sensitivities of eigenvalues to each element thickness were calculated in a similar manner. Fig. 2 shows the calculated sensitivities of each eigenvalue. In the figure, the horizontal axis represents the location of the element whose thickness is changed. The figure shows that the sensitivities fluctuate more as the mode number increases. When the thickness of the element near the fixed end increases, the eigenvalues for all modes increase. On the other hand, increasing of the thickness of the element at the tip decreases the eigenvalues for all modes.

Other parameters of the cantilever beam were changed and the sensitivities of eigenvalues to the element thickness were calculated. When the width b was doubled, the sensitivities did not change from the original case. When each of the thickness h, Young's modulus E, density ρ, and length l was doubled, the sensitivities became 2, 2, $\frac{1}{2}$, and $\frac{1}{16}$ times of the original case, respectively. From dimensional analysis the sensitivity of the eigenvalue λ_i of mode i can be expressed as follows.

$$\frac{\partial \lambda_i}{\partial h} = \frac{E}{\rho L^2} f(\frac{h}{L}) \tag{8}$$

The equation explains part of the above observations.

3.2 Simply supported beam

The sensitivities of eigenvalues to changes in the element thickness were calculated for a simply supported beam with the same material properties and dimensions as the previous cantilever beam. The beam was divided into 20 elements with equal length and the sensitivities of eigenvalues to each element thickness were calculated in a similar manner. Fig. 3 shows the calculated sensitivities of each eigenvalue. In the figure, the horizontal axis represents the location of the element whose thickness is changed. As expected, the sensitivities show symmetry. The figure shows that the sensitivities fluctuate more as the mode number increases, which is the same phenomenon as for a cantilever beam. The sensitivities are always positive for all modes. It means that increasing the thickness of any element results in increase of the eigenvalues of all modes.

4. Conclusions

The sensitivities of eigenvalues to the change of element thickness were calculated for beams in the paper. For a cantilever beam the sensitivities fluctuate more as the mode number increases. When the thickness of the element near the fixed end increases, the eigenvalues for all modes increase. On the other hand, increasing of the

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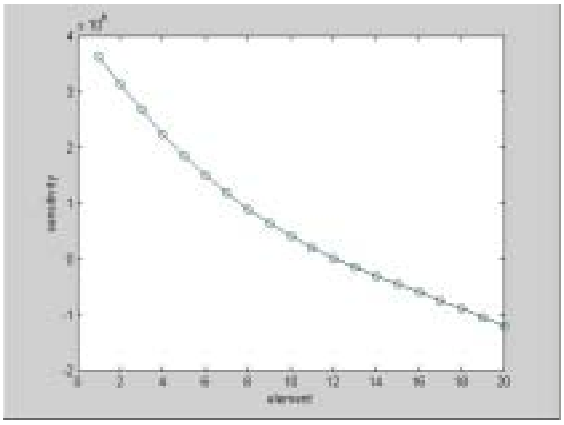
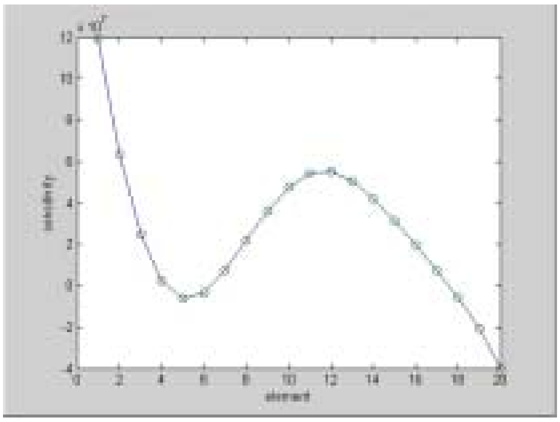
Table 2 Natural frequencies of the cantilever beam when the thickness of one element is increased by 10% (units : Hz)

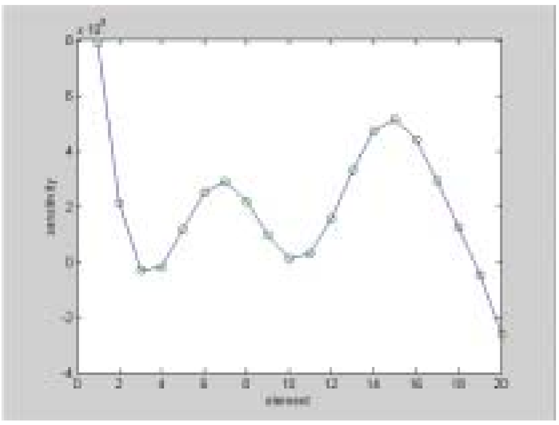
Position of element whose thickness is increased	None	1	2	3	4	5
f_1	15.7	17.0	16.3	15.8	15.5	15.2
f_2	98.4	102.5	98.7	101.7	100.8	97.3
f_3	276.4	282.9	282.1	278.0	287.8	276.8
f_4	546.0	556.8	552.9	560.6	556.9	553.1



(a) 1st eigenvalue

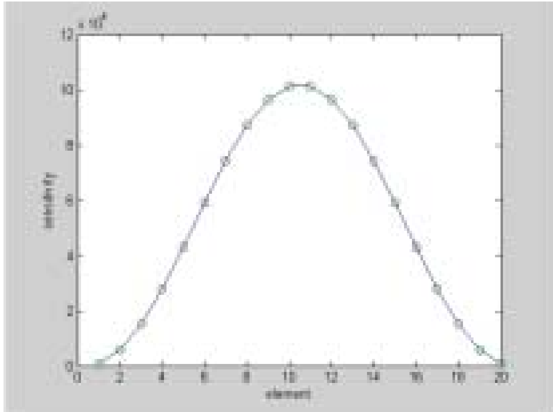


(b) 2nd eigenvalue

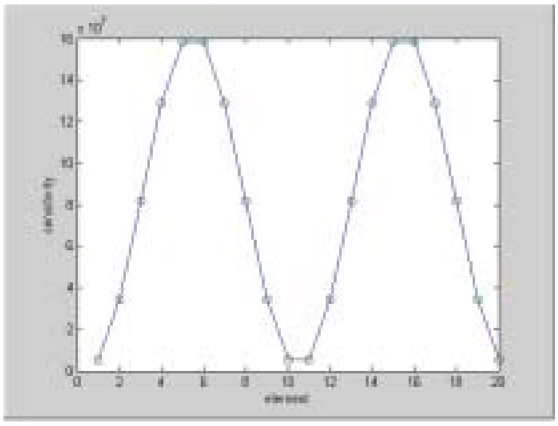


(b) 3rd eigenvalue

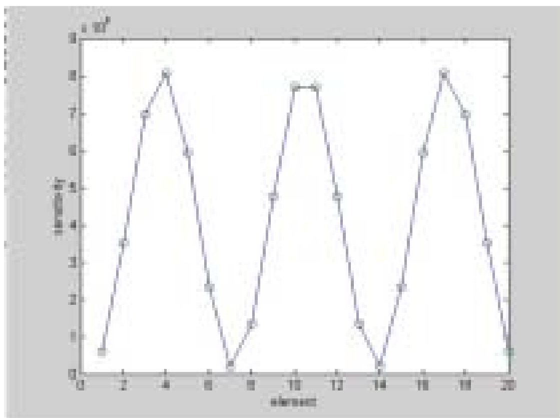
Fig. 2 Sensitivity of the eigenvalues to the element thickness for a cantilever beam



(a) 1st eigenvalue



(b) 2nd eigenvalue



(3) 3rd eigenvalue

Fig. 3 Sensitivity of the eigenvalues to the element thickness for a simply supported beam

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