A Bayesian Approach to Paired Comparison of Several Products of Poisson Rates

Dae-Hwang Kim[1], Hea-Jung Kim[2]

ABSTRACT

This article presents a multiple comparison ranking procedure for several products of the Poisson rates. A preference probability matrix that warrants the optimal comparison ranking is introduced. Using a Bayesian Monte Carlo method, we develop simulation-based procedure to estimate the matrix and obtain the optimal ranking via a row-sum scores method. Necessary theory and two illustrative examples are provided.

Key Words : Multiple comparison ranking; Noninformative prior; Preference probability matrix; Product of Poisson rates; Weighted Monte Carlo method;

1. INTRODUCTION

Suppose that $X_{ij} \sim P_o(\lambda_{ij})$ for $i=1,\ldots,K$ are K independent Poisson random variables with parameter λ_{ij} obtained from jth population, $j=1,\ldots,L$. Consider L independent populations each having single parameter λ_{j},; $j=1,\ldots,L$, with $K=1$ and suppose that there is an interest in the relative magnitudes of λ_j's. This is situation arising frequently in the paired comparison experimental arrangement. Various familiar methods and theories are possible for solving the problem of the interest. A great deal of work has focused on the problem of multiple inferences and simultaneous confidence intervals for λ_j's (Bauer (1997), Pennello (1997), Liu, Jamshidian, and Zhang (2004), and references therein). Bechhofer et al. (1995), Kim and Nelson (2001) developed selection methods for picking the best population in terms of magnitude of λ_j's. Unlike the previous works, of particular interest of the paper is the multiple comparison ranking of a scalar function $f(\delta_j)=\prod_{i=1}^{K}\lambda_{ij}, j=1,\ldots,L$, products of K Poisson rates, that may exhibit heterogeneity both in parameters, δ_j's, of interest and in the characteristic of the populations, where $\delta_j=(\lambda_{1j},\ldots,\lambda_{Kj})'$, $j=1,\ldots,L$.

Harris (1971) and Harris and Soms (1973) among others considered mathematical devices for the inference of $f(\delta_j)$. However, due to complex distributions involved in their devices, a multiple comparison among $f(\delta_j)$'s has not been seen in the literatures. Despite the fact

1) Institute of basic science, Yonsei university. Wonju. 220-710 daehha@empal.com
2) Department of Statistics Dongguk University Seoul 100-715, Korea kim3hj@dongguk.edu

that the problem of comparing $f(\delta_j)$'s may arise as a problem of interest in its own right, the comparison may be of more interest and will presumably be applied more often as approximate solutions to the problem of comparing the reliability of systems of K independent parallel components (see, for example, Mann et al. (1975)). This paper proposes a Bayesian procedure for the multiple ranking of $f(\delta_j)$'s and gives an application which has been selected for the purpose of motivating the contents of this article.

2. MULTIPLE COMPARISON RANKING METHOD

2.1. The Method

The testing of a set of objects for preference on specific criterion often requires of the researcher the ability to make very fine sensory discriminations based on complex distributional theory. To remove some of the complexity associated with simultaneously comparing several objects, the method of paired comparisons has been widely employed (see David (1987) and Davidson and Solomon (1973) and references therein). This section extends the method to the case of simultaneously comparing several parameter functions. Suppose L parameter functions, $f(\delta_j),; j=1,\ldots,L$, are to be compared in pairs with data obtained from L populations, and suppose a $L\times L$ matrix $\theta=\{\theta_{\alpha\beta}\};(\alpha,\beta=1,\ldots,L)$, where $\theta_{\alpha\beta}=\Pr(\alpha\rightarrow\beta)$ denotes the probability of preference for $f(\delta_\alpha)$ over $f(\delta_\beta)$, so that $\theta_{\alpha\beta}+\theta_{\beta\alpha}=1$. We take $\theta_{\alpha\alpha}=1/2$ for $\alpha=1,\ldots,L$. Given the preference probability matrix θ, one can obtain a rank order P of $f(\delta_j)'s$ where $P=(p_1,\ldots,p_L)$ is defined (in terms of preference order) as an arrangement of numbers in the index set $\{j|j=1,\ldots,L\}$ of $f(\delta_j)$'s such that p_α precedes p_β if $f(\delta_{p_\alpha})\rightarrow f(\delta_{p_\beta})$. If $v(P)$ denote the number of violations of the preference probability, that is the number of $\theta_{p_\alpha p_\beta}\geq 1/2$ in θ such that p_β precedes p_α in P, then following theorems enable us to obtain an optimal P.

DEFINITION 1. The preference probability matrix θ is said to have a strong stochastic transitivity property if its elements satisfy the condition given by

$$\theta_{\alpha\beta}\geq 1/2 \text{ implies } \theta_{\alpha\ell}\geq\theta_{\beta\ell} \text{ for } \alpha,\beta,\ell=1,\ldots,L, \tag{2.1}$$

or equivalently

$$\theta_{\alpha\beta}\geq 1/2 \text{ and } \theta_{\beta\ell}\geq 1/2 \text{ imply } \theta_{\alpha\ell}\geq \max(\theta_{\alpha\beta},\theta_{\beta\ell}). \tag{2.2}$$

Condition (2.2) follows directly from (2.1), while an examination of cases establishes that (2.1) follows from (2.2).

Suppose our interest is to rank a set of parameter functions, $f(\delta_j)$'s, where $f(\delta_j)$ denotes a scalar function of δ_j defined by jth probability model (or population) $\Pi_j, j=1,\ldots,L$, If a sample is available from each Π_j, then following theorem makes θ satisfy the strong stochastic transitivity condition.

THEOREM 1. Suppose paired comparison ranking is conducted according to descending order of magnitude in $f(\delta_j)$'s, and Suppose $\Theta=\{\theta_{\alpha\beta}\}$ is obtained from a proper joint posterior distribution of δ_j's such that

$$\theta_{\alpha\beta} = pr(f(\delta_\alpha) \rightarrow f(\delta_\beta)) = \frac{pr(\phi_{\alpha M} > 0 | Data)}{pr(\phi_{\alpha M} > 0 | Data) + pr(\phi_{\beta M} > 0 | Data)}, \quad (2.3)$$

where $\alpha, \beta = 1, \ldots, L$, $\phi_{\gamma M} = f(\delta_\gamma) - f_M$, $\gamma = \alpha, \beta$, and $f_M = \sum_{j=1}^{L} f(\delta_j)/L$. Then θ warrants the strong stochastic transitivity condition. Here $pr(A > 0 | Data)$ denotes the posterior probability of the interval $A > 0$.

THEOREM 2 (Kim, 2004a). Let $P = (p_1, \ldots, p_L)$ be a rank order of $f(\delta_j), j = 1, \ldots, L$ obtained from Θ satisfying the strong stochastic transitivity condition, then there is unique P and it has $v(P) = 0$. Moreover, P is equivalent to the ranking according to the magnitude of row-sum scores, w_j's, of Θ, where

$$W = (w_1, \ldots, w_L)' = \Theta \mathbf{1} \quad (2.4)$$

and $\mathbf{1}$ is the column vector of L ones.

3. A NONINFORMATIVE PRIOR FOR THE PRODUCT OF POISSON RATES

3.1. The Noninformative Prior

Suppose we observe X_{ij}, $i = 1, \ldots, K$, as independent Poisson variables with parameter λ_{ij} obtained from jth population Π_j, $j = 1, \ldots, L$. The parameter of interest being $f(\delta_j) = \prod_{i=1}^{K} \lambda_{ij}$, the product of K Poisson rates of Π_j. Given a parameter vector $\delta_j = (\lambda_{1j}, \ldots, \lambda_{Kj})'$, we seek a noninformative prior $\pi_s(\delta_j)$ so that the posterior interval for $f(\delta_j)$ has a coverage error of only $O_p(n^{-1})$ in the frequentist sense. Reasons for seeking this prior is described in Stein (1985) and he derived nonrigorously a sufficient condition for such a prior. Through the use of orthogonal parameters, Tibshirani (1989) gave a general form of the class of priors satisfying Stein's condition. Using Tibshirani's method, we see that the noninformative prior has the form given by

$$\pi_s(\delta_j) \propto g(\delta_j) \sqrt{\sum_{i=1}^{K} \lambda_{ij}^{-1}}, \quad \lambda_{ij} > 0, \; j = 1, \ldots, L, \quad (3.1)$$

where $\delta_j = (\lambda_{1j}, \ldots, \lambda_{Kj})'$ and $g(\delta_j) > 0$ is arbitrary. The derivation of (3.1) is the same as that for $\prod_{i=1}^{K} \lambda_{ij}^{\alpha_i}$ with $\alpha_i = 1$ given in Kim (2004b), and hence it is omitted.

The choice $g(\delta_j) = 1$ gives simple form of the prior which attains the asymptotic optimal frequentist coverage property. The class of prior given in (3.1) may be narrowed down to the second order probability matching priors as given in Mukerjee and Ghosh (1997). In contrast with the prior, the uniform prior (the Jeffreys prior), written π_u, is given by

$$\pi_u(\delta_j) \propto |I(\delta_j)|^{1/2} = \left(\prod_{i=1}^{K} \lambda_{ij} \right)^{-1/2}, \quad (3.2)$$

where $I(\delta_j)$ is the information matrix associated with the likelihood function. Note that π_s

is equivalent to π_u when $K=1$.

3.2. Comparison of The Priors

An appropriate noninformative prior should have good frequentist properties. Many authors (Mukerjee and Ghosh (1997); Datta, Ghosh and Mukerjee (2000); Mukerjee and Reid (1999) among others) suggested and argued those properties. One of them is that the frequentist coverage probability of a γ th posterior quantile should be close to γ. Using a WMC (weighted Monte Carlo) method, we investigate the property numerically for the priors π_s and π_u. The computation of the frequentist coverage probability of a γ th posterior quantile of $f(\delta)$ is based on the following algorithm. We suppress the index j in δ_j for convenience.

[Algorithm for Calculating the Frequentist Coverage Probability]

Step 1. Given a fixed true $\delta_0=(\lambda_{10},\ldots,\lambda_{k0})'$, simulate data x_i independently from $P_o(\lambda_{i0})$ distributions, $i=1,\ldots,K$.

Step 2. Obtain an importance sample of size n, $\{\delta^{(m)}=(\lambda_1^{(m)},\ldots,\lambda_k^{(m)})';m=1,2,\ldots,n\}$, from an importance function $g(\delta)$. Then calculate $f(\delta^{(m)})=\prod_{i=1}^{K}\lambda_i^{(m)}$ for $m=1,2,\ldots,n$.

Step 3. Sort $\{f(\delta^{(m)});\ m=1,2,\ldots,n\}$ to obtain the ordered values,

$$f(\delta_{(1)})\le f(\delta_{(2)})\le\cdots\le f(\delta_{(n)}).$$

Step 4. Compute the weighted function w_ℓ associated with ℓ th ordered value $f(\delta_{(\ell)})$. More specifically, we first compute

$$w_\ell=\frac{L(\delta^{(\ell)}|Data)\pi_0(\delta^{(\ell)})/g(\delta^{(\ell)})}{\sum_{m=1}^{n}L(\delta^{(m)}|Data)\pi_0(\delta^{(m)})/g(\delta^{(m)})}. \tag{3.3}$$

Then rewrite $\{w_\ell;\ \ell=1,2,\ldots,n\}$ as $\{w_{(\ell)};\ \ell=1,2,\ldots,n\}$ so that the ℓ th value $w_{(\ell)}$ corresponds to the ℓ th value $f(\delta_{(\ell)})$.

Step 5. Calculate

$$\rho=\begin{cases}0 & \text{if } f(\delta_0)<f(\delta_{(1)}),\\ \sum_{\ell=1}^{m}w_{(\ell)} & \text{if } f(\delta_{(m)})\le f(\delta_0)<f(\delta_{(m+1)}),\\ 1 & \text{if } f(\delta_0)\ge f(\delta_{(n)}),\end{cases} \tag{3.4}$$

where $f(\delta_0)=\prod_{i=1}^{K}\lambda_{i0}$.

Repeat Step 1 and Step 5 with n^* times, and compute the proportion τ of $\rho\le\gamma$ in these replications.

When the uniform prior $\pi_u(\delta)$ is used, the posterior distribution is a kernel of the joint density of k independently distributed $\Gamma(x_i+1/2,1)$ variates:

$$\pi(\delta|data)\propto\prod_{i=1}^{K}\lambda_i^{x_i-1/2}\exp\{-\lambda_i\},\quad \lambda_i>0,\ i=1,\ldots,k.$$

Thus the MC sample in Step 2 is directly generated from the posterior distribution

$\pi(\delta|Data)$ giving $w_\ell = 1/n$.

When the noninformative prior $\pi_s(\delta)$ is used, the joint posterior distribution is

$$\pi(\delta|data) \propto \left(\sum_{i=1}^{K} \lambda_i^{-1}\right)^{1/2} \prod_{i=1}^{K} \lambda_i^{x_i} \exp\{-\lambda_i\}, \quad \lambda_i > 0; i = 1, \ldots, K.$$

The algorithm needs to specify the importance function $g(\delta)$. The most natural candidate for the importance distribution is the joint distribution of independent $\Gamma(x_i+1, 1)$, $i = 1, \ldots, K$, variates yielding the weights

$$w_\ell = \left(\sum_{i=1}^{K} 1/\lambda_i^{(\ell)}\right)^{1/2} / \sum_{m=1}^{n}\left(\sum_{i=1}^{K} 1/\lambda_i^{(m)}\right)^{1/2}, \quad \ell = 1, 2, \ldots, n. \tag{3.5}$$

The quantity ρ is the estimate of the marginal posterior probability of $f(\delta)$ for the interval $(0, f(\delta_0))$. On the other hand τ is the estimated frequentist coverage probability of the γ th posterior quantile. Table 1 shows the estimated frequentist coverage probabilities of $\gamma = .05(.95)$th posterior quantiles of $f(\delta) = \prod_{i=1}^{K} \lambda_i$ for various true values, δ_0's, of δ obtained by using π_s and π_u when $K=4$.

TABLE 1. Frequentist Coverage Probabilities for .05(.95)th Posterior Quantiles of $f(\delta)$.

$K=4$						
δ_0	(1,2,3,4)	(1,2,5,5)	(5,5,5,5)	(5,5,6,7)	(5,5,10,10)	(10,10,10,10)
π_s	.052(.978)	.052(.979)	.054(.949)	.051(.950)	.051(.947)	.049(.945)
π_u	.022(.901)	.026(.914)	.028(.899)	.028(.900)	.031(.908)	.032(.913)

For the calculations of the entries in the table, n is 10,000 and n^* is 10,000. The maximum standard errors of estimations ρ and δ are .0042 and .006 respectively.
From Table 1, we see that the frequentist coverage probabilities obtained from using the noninformative prior π_s are almost close to the desired levels, while those obtained from using π_u underestimates the levels. Table 1 also notes that small-sample ($K=4$) frequentist coverage probabilities of π_s are uniformly better than those for π_u in all the situations. Therefore, the Tibshirani's asymptotically optimal frequentist coverage prior π_s is more appealing in the sense of the frequentist property.

4. BAYESIAN ESTIMATION OF THE PREFERENCE MATRIX

4.1. A WMC Method for Calculating the Preference Probability Matrix

In this section we recover the suppressed index $j, j=1, \ldots, L$, denoting jth probability model (or population), to compute the preference probability defined in (2.3).
Using the noninformative priors $\pi_s(\delta_j)$, we obtained the joint posterior distribution of δ_j's:

$$\pi(\delta_1, \ldots, \delta_L|Data) \propto \prod_{j=1}^{L}\left(\sum_{i=1}^{K} \lambda_{ij}^{-1}\right)^{1/2} \prod_{i=1}^{K} \lambda_{ij}^{x_{ij}} \exp\{-\lambda_{ij}\}, \quad \lambda_{ij} > 0. \tag{4.1}$$

Suppose paired comparison ranking is conducted according to descending order of magnitude in $f(\delta_j)$'s. Then we need to estimate $\theta=\{\theta_{\alpha\beta}\}$ from (4.1) where the preference probabilities are defined by (2.3). It is seen, from Theorem 1, that θ satisfies the strong stochastic transitivity condition so that the paired comparison ranking is simply obtained from the row-sum scores in (2.4). The posterior probability $pr(\phi_{\gamma M}>0|Data)$ in (2.4) can be expressed as an integral-type posterior quantity of (4.1) so that

$$E[h_\gamma(\delta_1,\ldots,\delta_L)|Data)]=\int I(f(\delta_\gamma)-f_M>0)\ \pi(\delta_1,\ldots,\delta_L|Data)\partial\delta_1\ldots,\partial\delta_L, \tag{4.2}$$

where $I(f(\delta_\gamma)-f_M>0)$, $\gamma=1,\ldots,L$, denotes the indicator function. It is seen, from (4.1), that analytical evaluation of the joint posterior distribution for $f(\delta_j)$'s does not appear possible. Thus exact calculation of the preference probabilities, $\theta_{\alpha\beta}$'s, is infeasible. Furthermore, as defined in (2.3), $\theta_{\alpha\beta}$ is a complex function of $f(\delta_j)$'s. Thus we need a special computational scheme. We will describe how to apply a weighted Monte Carlo (WMC) method to overcome these difficulties arising in the paired comparison ranking of $f(\delta_j)$'s. The method needs to specify an importance function $g(\delta_1,\ldots,\delta_L)$ to get the WMC estimator of (4.2) with an importance sampling scheme: Let an importance sample of size n be $\{(\delta_1^{(m)},\delta_2^{(m)},\ldots,\delta_L^{(m)}),m=1,2,\ldots,n\}$, from the importance function $g(\delta_1,\ldots,\delta_L)$, where $\delta_j^{(m)}=(\lambda_{1j}^{(m)},\ldots,\lambda_{Kj}^{(m)})',j=1,\ldots,L$. Then a WMC estimator of (4.2) is

$$\widehat{E}[h_\gamma(\delta_1,\ldots,\delta_L)|Data)]=\frac{\sum_{m=1}^{n} w_m I(f(\delta_\gamma^{(m)})-f_M^{(m)}>0|Data)}{\sum_{\ell=1}^{n} w_\ell}, \tag{4.3}$$

where

$$w_m=\prod_{j=1}^{L} L(\delta_j^{(m)}|Data)\pi_s(\delta_j^{(m)})/g(\delta_1^{(m)},\ldots,\delta_L^{(m)}) \tag{4.4}$$

is the importance sampling weight and $f_M^{(m)}=\sum_{j=1}^{L} f(\delta_j^{(m)})/L$. The most natural candidate for the importance distribution is the joint distribution of independent $\Gamma(x_{ij}+1,1)$, $i=1,\ldots,K;\ j=1,\ldots,L$ variates yielding the weights

$$w_m=\prod_{j=1}^{L}\left(\sum_{i=1}^{K}1/\lambda_{ij}^{(m)}\right)^{1/2},\quad m=1,2,\ldots,n. \tag{4.5}$$

Using (4.3), we obtain an estimator of the desired preference probability in (2.3):

$$\widehat{\theta}_{\alpha\beta}=\frac{\widehat{E}[h_\alpha(\delta_1,\ldots,\delta_L)|Data)]}{\widehat{E}[h_\alpha(\delta_1,\ldots,\delta_L)|Data)]+\widehat{E}[h_\beta(\delta_1,\ldots,\delta_L)|Data)]}, \tag{4.6}$$

and hence the estimator of the preference matrix, $\widehat{\Theta}$.

Geweke (1989) showed that

$$\widehat{E}[h_\gamma(\delta_1,\ldots,\delta_L)|Data)]\rightarrow E[h_\gamma(\delta_1,\ldots,\delta_L)|Data)],\ \ a.s.\ \ as\ \ n\rightarrow\infty.$$

This implies that $\widehat{E}[h_\gamma(\delta_1,\ldots,\delta_L)|Data)]$ is consistent estimator of $E[h_\gamma(\delta_1,\ldots,\delta_L)|Data)]$, and hence $\widehat{\theta}_{\alpha\beta}$ is also consistent estimator of $\theta_{\alpha\beta}$ by the properties of the consistent estimators. Therefore, the paired comparison ranking among L products of the Poisson rates can be conducted by the row-sum scores method based on the posterior estimator $\hat{\theta}$ obtained from the suggested WMC method.

5. ILLUSTRATIVE EXAMPLES

The suggested ranking procedure using $\pi_s(\delta_j)$ is applied to a real data set. The data is selected for the purpose of motivating the contents of this paper. The data is obtained from Mann et al. (1974, P.498). They used it for calculating nonrandomized lower confidence bounds on reliability for a series system of three components. Table 3 lists the data where x_{ij}, n_{ij}, and γ_{ij} denote, respectively, the number of failures, the number of experiments at risk, and observed failure rate obtained form independent Bernoulli trials conducted on ith component in jth system ($i=1,2,3$; $j=1,2,3,4,5$). Thus respective probabilities of system failure are $\psi_j = p_{1j}p_{2j}p_{3j}$, $j=1,\ldots,5$ products of three binomial parameters. The straightforward modification of the paired-comparison ranking procedure for $f(\delta_j)$'s gives the multiple comparison ranking for ψ_j's. Assuming that Poisson approximation to the binomial distributions is satisfactory to the user, the modification simply needs to change with $\psi_j = f(\delta_j)/\prod_{i=1}^{K} n_{ij}$ in (2.3). Using the suggested WMC method with the importance sample of size $n=10,000$ (described in Section 4), we estimated the preference probability matrix $\Theta = \{\theta_{\alpha\beta}\}, \alpha, \beta = 1,\ldots,5$, for ranking ψ_j's. The estimated preference probability matrix is given by

$$\hat{\Theta} = \begin{pmatrix} .500 & .585 & .593 & .512 & .498 \\ .415 & .500 & .509 & .428 & .414 \\ .407 & .491 & .500 & .419 & .405 \\ .488 & .572 & .581 & .500 & .486 \\ .502 & .586 & .595 & .514 & .500 \end{pmatrix}$$

TABLE 2. The Data Set and Row-Sum Scores

System j	Number of Experiments (n_{1j}, n_{2j}, n_{3j})	Observed Failures (x_{1j}, x_{2j}, x_{3j})	Failure Rates $(\gamma_{1j}, \gamma_{2j}, \gamma_{3j})$	Row-Sum Scores
System 1	(10, 10, 9)	(0, 1, 1)	(0, 1/10, 1/9)	2.688
System 2	(20, 20, 20)	(1, 1, 1)	(1/20, 1/20, 1/20)	2.266
System 3	(30, 30, 30)	(1, 2, 3)	(1/30, 1/15, 1/10)	2.222
System 4	(50, 50, 50)	(1, 2, 4)	(1/50, 1/25, 2/25)	2.627
System 5	(100, 100, 100)	(2, 3, 5)	(1/50, 3/100, 1/20)	2.697

Calculation of $\hat{\Theta}\mathbf{1}$ gives the row-sum scores. Resulting row-sum score of each system is also tabulated in Table 2. According to the row-sum scores, we see that the optimal paired comparison ranking of five systems is $P(5,1,4,2,3)$ with $v(P)=0$. Thus the order of failure probabilities among the five systems is $\psi_5 < \psi_1 < \psi_4 < \psi_2 < \psi_3$, and hence we can conclude that the fifth system is the most reliable system among them.

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