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Pricing Outside Barrier Options

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Abstract

This paper will derive explicit unified pricing formulas for eight types of outside barrier options, respectively. The monitoring periods of these options start at an arbitrary date and end at another arbitrary date before maturity. The eight types of barrier options are up-and-in, up-and-out, down-and-in and down-and-out call (or put) options.

Key words: barrier options, Brownian motion, Esscher transform

1. Introduction

The payoffs of outside barrier options depend on prices of two underlying assets: one asset, called the payoff asset, is used for determining the payoff, and the other asset, called the barrier asset, for determining whether the options knock in or out. Outside barrier options are designed for investors who anticipate that the barrier asset will hit a barrier, and are cheaper than the corresponding inside barrier options under some assumptions of the correlation between the payoff and barrier assets.

Heynen and Kat (1994) derived pricing formulas for outside barrier options whose monitoring period is $[0, T]$. Bermin (1996) developed explicit pricing formulas for outside barrier options with the monitoring period from time 0 to time t ($t < T$). This paper will derive an explicit unified pricing formula for outside barrier options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity.

2. Esscher Transforms and Some Distributions

This section discusses some basics for pricing contingent claims and derives some useful distribution functions for pricing barrier options. If we assume the Black-Scholes

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framework, then according to the fundamental theorem of asset pricing, the prices of contingent claims such as options can be calculated as the discounted expectations of the corresponding payoffs with respect to the equivalent martingale measure. Gerber and Shiu (1994, 1996) showed that Esscher transforms are an efficient tool for finding the equivalent martingale measure.

Let $S_1(t)$ and $S_2(t)$ denote the time-t prices of two underlying assets. Assume that these assets pay no dividends. Assume that for $t \geq 0$, $i = 1$ and 2,

$$S_i(t) = S_i(0)\exp(X_i(t)), \tag{2.1}$$

where $\{\boldsymbol{X}(t) = (X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $X_i(0) = 0$ and diffusion matrix $\boldsymbol{V}$ equal to

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{2.2}$$

Thus the 2-dimensional Brownian motion is a stochastic process with independent and stationary increments and $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu}t$ and covariance matrix $\boldsymbol{V}t$.

For a nonzero real vector $\boldsymbol{h} = (h_1, h_2)'$, the moment generating function of $\boldsymbol{X}(t)$, $\mathrm{E}[e^{\boldsymbol{h}\boldsymbol{X}(t)}]$, exists for all $t \geq 0$, because $\{\boldsymbol{X}(t)\}$ is the Brownian motion as described above. The stochastic process

$$\{e^{\boldsymbol{h}'\boldsymbol{X}(t)}\mathrm{E}[e^{\boldsymbol{h}'\boldsymbol{X}(1)}]^{-t}\} \tag{2.3}$$

is a positive martingale which can be used to define a new probability measure Q. For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$, the expectation of Y under the new probability measure Q is calculated as

$$\mathrm{E}[Y\frac{e^{\boldsymbol{h}'\boldsymbol{X}(T)}}{\mathrm{E}[e^{\boldsymbol{h}'\boldsymbol{X}(1)}]^T}], \tag{2.4}$$

which will be denoted by $\mathrm{E}[Y; \boldsymbol{h}]$. The risk-neutral measure is the Esscher measure of parameter vector $\boldsymbol{h} = \boldsymbol{h}^*$ with respect to which the process $\{e^{-rt}S_i(t)\}$ is a martingale. Thus

$$\mathrm{E}[e^{-rt}S_i(t); \boldsymbol{h}^*] = S_i(0). \tag{2.5}$$

Therefore, $\boldsymbol{h}^*$ is the solution of

$$\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h}^* = (r - \sigma_1^2/2,\, r - \sigma_2^2/2)'. \tag{2.6}$$

For $t \geq 0$, the moment generating function of $\boldsymbol{X}(t)$ under Esscher measure of parameter vector $\boldsymbol{h}$ is

$$\mathrm{E}[e^{z'X(t)}; \boldsymbol{h}] = \exp\{(\boldsymbol{\mu}' + \boldsymbol{h}'\boldsymbol{V})zt + z'Vzt/2\}, \tag{2.7}$$

which implies that $\boldsymbol{X}(t)$ has a bivariate normal distribution with mean vector $(\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h})t$ and variance $\boldsymbol{V}t$ under the Esscher measure. It can be shown that the process $\{\boldsymbol{X}(t)\}$ under the Esscher measure has independent and stationary increments. Thus, this process is a two-dimensional Brownian motion with drift vector

$$\boldsymbol{\mu} + \boldsymbol{V}\boldsymbol{h} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{2.8}$$

and diffusion matrix $\boldsymbol{V}$ under the Esscher measure of parameter vector $\boldsymbol{h}$.

Let us consider a special case of the factorization formula (Gerber and Shiu, 1994, 1996). For a random variable Y that is a real-valued function of $\{\boldsymbol{X}(t), 0 \leq t \leq T\}$,

$$\mathrm{E}[e^{c'X(T)}\, Y; \boldsymbol{h}] = \mathrm{E}[e^{c'X(T)}; \boldsymbol{h}]\mathrm{E}[Y; \boldsymbol{h} + \boldsymbol{c}]. \tag{2.9}$$

Now, let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of the random vector $\boldsymbol{Z}$ is

$$\Phi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = \Pr(Z_1 \leq a, Z_2 \leq b, Z_3 \leq c). \tag{2.10}$$

Finally, let us consider a two-dimensional Brownian motion $\{\boldsymbol{X}(t) = (X_1(t), X_2(t))'\}$. For $0 \leq s \leq t$, let

$$M_2(s, t) = \max\{X_2(\tau), s \leq \tau \leq t\} \tag{2.11}$$

be the maximum of the Brownian motion $\{X_2(\tau), 0 \leq \tau\}$ between time s and time t. It can be proved in Lee (2004) that for $0 < s < t \leq T$, the joint distribution function of $M_2(s, t)$ and $X_1(T)$ is

$$\Pr(X_1(T) \leq x, M_2(s, t) \leq m)$$

$$= \Phi_3(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m-\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}})$$

$$-e^{\frac{2\mu_2}{\sigma_2^2}m} \Phi_3(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, \frac{-m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m+\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}). \quad (2.12)$$

3. Outside Barrier Options

Let us take a look at the activating conditions of the outside barrier options: up or down, in or out and call or put. Assume that the strike price is K, and the barrier level is B. Let $b = \log[B/S_2(0)]$ and $k = \log[K/S_1(0)]$. The activating condition of the up-and-in (or up-and-out) barrier option is $\{M_2(s, t) > b\}$ (or $\{M_2(s, t) < b\}$), while that of the down-and-in (or down-and-out) option is $\{m_2(s, t) < b\}$ (or $\{m_2(s, t) > b\}$). The call (or put) condition is $\{X_1(T) > k\}$ (or $\{X_1(T) < k\}$). The payoff of the call (or put) option will be $(S_1(T) - K)$ (or $(K - S_1(T))$) if the option satisfies its activating condition and $S_1(T)$ is greater (or less) than K.

Let $A_{\alpha,\beta}^{outside}$ denote the activating conditions where $\alpha = 1$ or -1 implies up or down and $\beta = 1$ or -1 implies in or out. Thus the activating condition $A_{1,\beta}^{outside}$ denotes the event $\{\beta M_2(s, t) > \beta b\}$ for up barrier options and $A_{-1,\beta}^{outside}$ denotes $\{\beta m_2(s, t) < \beta b\}$ for down barrier options. Let $\gamma = 1$ and -1 represent call and put options, respectively. For example, $\alpha = -1$, $\beta = 1$ and $\gamma = -1$ represent the down-and-in outside barrier put option. The payoff for the eight types of outside barrier options can be expressed in a unified way as follows:

$$\gamma(S_1(T) - K), \text{ if } A_{\alpha,\beta}^{outside} \text{ and } \gamma X_1(T) > \gamma k$$

$$0, \text{ otherwise,} \quad (3.1)$$

that is,

$$\gamma(S_1(T) - K)\mathrm{I}(A_{\alpha,\beta}^{outside}, \gamma X_1(T) > \gamma k). \quad (3.2)$$

By the fundamental theorem of asset pricing, the time-0 value of the payoff (3.9) is

$$e^{-rT}\mathrm{E}[\gamma(S_1(T) - K)\mathrm{I}(A_{\alpha,\beta}^{outside}, \gamma X_1(T) > \gamma k); \boldsymbol{h}^*]$$

$$= \gamma\{S_1(0)e^{-rT}\mathrm{E}[e^{X_1(T)}\mathrm{I}(A_{\alpha,\beta}^{outside}, \gamma X_1(T) > \gamma k); \boldsymbol{h}^*]$$

$$- e^{-rT}K\Pr(A_{\alpha,\beta}^{outside},\ \gamma X_1(T) > \gamma k;\ \boldsymbol{h}^*)\}. \tag{3.3}$$

Applying the factorization formula (2.9) to the expectation on the right-hand side of (3.3), the time-0 value of the outside barrier options is

$$\gamma\{S_1(0)\Pr(A_{\alpha,\beta}^{outside},\ \gamma X_1(T) > \gamma k;\ \boldsymbol{h}^*+\mathbf{1}_1) - e^{-rT}K\Pr(A_{\alpha,\beta}^{outside},\ \gamma X_1(T) > \gamma k;\ \boldsymbol{h}^*)\}, \tag{3.4}$$

where $\mathbf{1}_1$ denotes $(1, 0)'$.

Now, the final step for pricing these outside barrier options is to calculate the probabilities of (3.4). These probabilities are the same except that the drift parameter vectors of the first and second probabilities are

$$(\mu_1^{**}, \mu_2^{**}) = (r + \sigma_1^2/2,\ r - \sigma_2^2/2 + \rho\sigma_1\sigma_2) \tag{3.5a}$$

and

$$(\mu_1^{*}, \mu_2^{*}) = (r - \sigma_1^2/2,\ r - \sigma_2^2/2), \tag{3.6b}$$

respectively. Let

$$P_{\alpha,\beta,\gamma}(\mu_1, \mu_2, \rho, b, k) := \Pr(A_{\alpha,\beta}^{outside},\ \gamma X_1(T) > \gamma k). \tag{3.7}$$

The following formula (3.8) is a unified expression for the eight probabilities of the events $\{A_{\alpha,\beta}^{outside},\ \gamma X_1(T) > \gamma k\}$. It can be shown that the probability

$$\begin{aligned} &P_{\alpha,\beta,\gamma}(\mu_1, \mu_2, \rho, b, k) \\ &= \frac{\beta+1}{2}\Phi(-\gamma d_{11}^*) - \beta\{\Phi_3(-\gamma d_{11}^*, \alpha d_{12}^*, \alpha d_{13}^*; -\alpha\gamma\rho\sqrt{\frac{t}{T}}, -\alpha\gamma\rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}) \\ &\quad - e^{\frac{2\mu_2}{\sigma^2}b}\Phi_3(-\gamma d_{21}^*, \alpha d_{22}^*, \alpha d_{23}^*; -\alpha\gamma\rho\sqrt{\frac{t}{T}}, \alpha\gamma\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}})\}, \end{aligned} \tag{3.8}$$

where

$$d_{11}^* = \frac{k-\mu_1 T}{\sigma_1\sqrt{T}},\quad d_{12}^* = \frac{b-\mu_2 t}{\sigma_2\sqrt{t}},\quad d_{13}^* = \frac{b-\mu_2 s}{\sigma_2\sqrt{s}},$$

$$d_{21}^* = \frac{k-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho b}{\sigma_2\sqrt{T}},\quad d_{22}^* = \frac{-b-\mu_2 t}{\sigma_2\sqrt{t}},\quad d_{23}^* = \frac{b+\mu_2 s}{\sigma_2\sqrt{s}}.$$

Thus, applying the probability (3.8) to (3.4), we will obtain the time-0 value of the barrier options,

$$\gamma[S(0)\, P_{\alpha,\beta,\gamma}(\mu_1^{**}, \mu_2^{**}, \rho, b, k) - e^{-rT} K\, P_{\alpha,\beta,\gamma}(\mu_1^{*}, \mu_2^{*}, \rho, b, k)]. \tag{3.9}$$

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