$P_{\lambda,\tau}^{M}$-policy of a finite dam with both continuous and jumpwise inputs

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Abstract

A finite dam under $P_{\lambda,\tau}^{M}$-policy is considered, where the input of water is formed by a Wiener process subject to random jumps arriving according to a Poisson process. Explicit expression is deduced for the stationary distribution of the level of water. And the long-run average cost per unit time is obtained after assigning costs to the changes of release rate, a reward to each unit of output, and a penalty which is a function of the level of water in the reservoir.

Keywords: finite dam, $P_{\lambda,\tau}^{M}$-Policy, Wiener process, compound Poisson process, stationary distribution, long-run average cost.

1. Introduction

Since Faddy(1974) introduced a P_{λ}^{M}-policy to a finite dam with input formed by a Wiener process, the model has been generalized in various ways by many authors. Yeh(1985) and Yeh and Hua(1987) introduced a more general policy, $P_{\lambda,\tau}^{M}$-policy, to the finite dam with input of Wiener process and obtained the long-run average cost per unit time after assigning costs to the changes of releasing rate, a reward to each unit of output, and a penalty depending on the level of water. Lee and Ahn(1998) applied the P_{λ}^{M}-policy to an infinite dam with input formed by a compound Poisson process. Abdel-Hameed(2000) studied the $P_{\lambda,\tau}^{M}$-policy in the infinite dam where the input process is a compound Poisson process with positive drift. Bae, Kim and Lee(2003) generalized Abdel-Hameed's model to the case of finite dam when the input is formed by a compound Poisson process and the level of water between inputs decreases linearly at a constant rate. Bae, Kim and Lee(2003) obtained the long-run average cost per unit time after assigning the same costs to the dam as Yeh(1985) did.

We, in this paper, consider a finite dam under $P_{\lambda,\tau}^{M}$-policy where the input process is a Wiener process subject to compound Poisson jumps. The level of water is initially at 0 and thereafter follows a Wiener process with drift μ ($-\infty < \mu < \infty$), variance $\sigma^2 > 0$, and reflecting barriers at both 0 and V, where V is the capacity of the reservoir. Meanwhile, the level of water also increases jumpwise due to the instantaneous inputs such as rains which occur according to a Poisson process with rate $\nu > 0$. The amounts of instantaneous inputs are independent and identically distributed with distribution function G and mean m. If the level of water exceeds V after an instantaneous

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input, we assume that the exceeding amount of water over V overflows immediately so that the level of water becomes V. We also assume that $\nu m + \mu > 0$ so that the level of water eventually increases. At the moment when the level of water crosses a threshold λ $(0 < \lambda < V)$ over, we start to release water at a constant rate $M > 0$. Note that the level of water now follows the Wiener process of drift $\mu - M$ and variance σ^2, which still have 0 and V as reflecting barriers and are subject to compound Poisson jumps. It continues to release water until the level of water reaches τ $(0 < \tau < \lambda)$, and at this moment we stop releasing water. Thereafter, we wait until the level of water exceeds the threshold λ again.

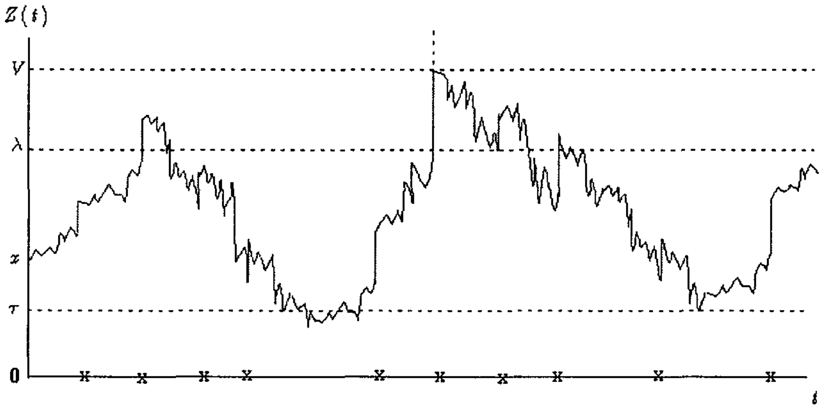

Figure 1: $\{Z(t), t \geq 0\}$

Let $\{Z(t), t \geq 0\}$ be the level of water at time t in our model. A sample path of $Z(t)$ is shown in Figure 1. To obtain the stationary distribution of $\{Z(t), t \geq 0\}$, we divide the process $\{Z(t), t \geq 0\}$ into the following two processes: Process $\{Z_0(t), t \geq 0\}$ is formed by separating from the original process the periods of releasing rate being 0 and by connecting them together. Process $\{Z_M(t), t \geq 0\}$ is formed by connecting the rest of the original process together.

2. Stationary distribution of Z(t)

Let T_0 and T_M denote in a cycle the periods of releasing rate being 0 and M, respectively. That is, T_0 is the period from a renewal point to the point where $Z(t)$ crosses λ over for the first time and T_M is the period from the latter point to the next renewal point. We also denote the process of the level of water during T_0 by $Z_0(t)$ and the process during T_M by $Z_M(t)$.
Let $F_1(x,t) = P\{Z_0(t) \leq x\}$ and $F_2(x,t) = P\{Z_M(t) \leq x\}$ denote the distribution of $Z_0(t)$ and $Z_M(t)$, respectively. Note that $\{Z_0(t), t \geq 0\}$ and $\{Z_M(t), t \geq 0\}$ are a regenerative process. Define

$$w(x_0) = E\Big[\int_0^{T_0} h(Z_0(t))dt | Z_0(0) = x_0\Big], \quad \text{for} \quad 0 \leq x_0 \leq \lambda$$

and

$$u(x_0) = E\Big[\int_0^{T_M} h(Z_M(t))dt | Z_M(0) = x_0\Big], \quad \text{for} \quad \tau \le x_0 \le V,$$

with a penalty function $h(u)$ is assigned to the dam per unit time when $Z(t) = z(0 \le z \le V)$. Then, $E(T_0)$ and $E(T_M)$ can be easily drive from $w(\tau)$ and $E[u(Z_0(T_0))|Z_0(0) = \tau]$ by putting $h(u) \equiv 1$. Also, we define

$$h(u) = \begin{cases} 1, & u \le x \\ 0, & \text{otherwise.} \end{cases}$$

Then, $w(\tau)$ and $E[u(Z_0(T_0))|Z_0(0) = \tau]$ are the expected period where $Z_1(t)$ and $Z_2(t)$ are less than or equal to x during T_0 and T_M. Since $\{Z_1(t), t \ge 0\}$ $\{Z_2(t), t \ge 0\}$ are a regenerative process, the stationary distribution of $Z_1(t)$ and $Z_2(t)$ are given by

$$F_1(x) = \lim_{t\to\infty} F_1(x,t) = \frac{w(x,\tau)}{E[T_0]}$$

and

$$F_2(x) = \lim_{t\to\infty} F_2(x,t) = \frac{E[u(Z_0(T_0))|Z_0(0) = \tau]}{E[T_M]}.$$

Let T^* be the generic random variable denoting the time between successive renewals. Then

$$T^* = T_0 + T_M. \tag{1}$$

Proposition. $F(x)$ is given by the following weighted average of $F_1(x)$ and $F_2(x)$:

$$\frac{E(T_0)}{E(T^*)} F_1(x) + \frac{E(T_M)}{E(T^*)} F_2(x). \tag{2}$$

3. Long-run average cost per unit time

We assign four costs to the dam. $K_1 M$ is the cost of changing releasing rate from 0 to M and $K_2 M$ the cost of changing releasing rate from M to 0. A reward is given to each unit of output while the water being released. Consider the points where $Z(t)$ crosses τ down for the first time after we start to release water. These are the points where we close the gate of the dam. Note that the sequence of these points forms an embedded delayed renewal process. Note that in the model of Yeh(1985) the level of water crosses λ over always through a continuous path and in Bae, Kim and Lee(2003) always by jump. In our model, however, $Z(t)$ crosses λ over in both ways.

By making use of the renewal reward theorem[Ross(1983, p.78)], we can see that the long-run average cost per unit time is given by

$$\begin{aligned} C(M,\lambda,\tau) &= \frac{E[\text{cost during a cycle}]}{E[\text{length of a cycle}]} \\ &= \frac{(K_1 + K_2)M + w(\tau) - ME[T_M] + E[u(Z_0(T_0))|Z_0(0) = \tau]}{E(T_0) + E(T_M)}. \end{aligned} \tag{3}$$

4. Evaluations

In this section, we evaluate the functions $w(x)$ for all $0 \le x \le \lambda$, $u(x)$ for all $\tau \le x \le V$ and the distribution of $Z_0(T_0)$ given that $Z_0(0) = x$ for all $0 \le x \le \lambda$ by establishing backward differential equations and converting the equations into renewal type equations. Then, $w(\tau)$ is nothing but $w(x)|_{x=\tau}$ and $E[u(Z_0(T_0))|Z_0(0) = \tau]$ is easily obtained from $u(x)$ by conditioning on $Z_0(T_0)$ while setting $x = \tau$.

Let $B_0(t)$ and $B_M(t)$ are Wiener process with drift μ and $\mu - M$, variance σ^2, and reflecting barrier 0 and V, respectively. Let Δ_0 and Δ_M are the increment of $B_0(t)$ and $B_M(t)$ in an interval of length h, respectively.

4.1 w(x), for $0 \le x \le \lambda$

To evaluate $w(x)$, we first need to show that $w(x)$ satisfies the boundary conditions :

Lemma 1. $w(\lambda) = 0$ and $w'(0) = 0$.

We now derive the backward differential equation for $w(x)$. Suppose that $Z_0(0) = x$, $0 \le x \le \lambda$. Conditioning on whether a jump occurs or not during $[0, h]$ gives that

$$w(x) = \begin{cases} E[\int_0^h f(B_0(t))dt + w(x+\Delta_0)], & \text{if no jump occurs} \\ E[\int_0^h f(B_0(t))dt], & \text{if a jump occurs and } Y \ge \lambda - x - \Delta_0 \\ E[\int_0^h f(B_0(t))dt + w(x+\Delta_0+Y)], & \text{if a jump occurs and } Y < \lambda - x - \Delta_0. \end{cases}$$

Hence, we have

$$\begin{aligned} w(x) &= (1-\nu h)E\left[\int_0^h f(B_0(t))dt + w(x+\Delta_0)\right] + o(h) \\ &+\nu h E\left[\int_0^h f(B_0(t))dt | x+Y+\Delta_0 \ge \lambda\right] Pr\{x+Y+\Delta_0 \ge \lambda\} \\ &+\nu h E\left[\int_0^h f(B_0(t))dt + w(x+\Delta_0+Y) | x+Y+\Delta_0 < \lambda\right] Pr\{x+Y+\Delta_0 < \lambda\}. \end{aligned}$$

Taking Taylor series expansion on $w(x+\Delta_0)$, rearranging the equation and letting $h \to 0$ yield

$$0 = f(x) + \mu w'(x) + \frac{\sigma^2}{2} w''(x) - \nu w(x) + \nu \int_0^{\lambda - x} w(x+y)dG(y). \tag{4}$$

For the convenience of analysis, we define $\bar{w}(x) = w(\lambda - x)$. Then, $\bar{w}(x)$ satisfies the following renewal type equation:

Lemma 2.

$$\bar{w}(x) = \bar{w}'(0)x - \frac{2}{\sigma^2}\int_0^x F_\lambda(t)dt + \int_0^x \bar{w}(x-t)dW(t) \tag{5}$$

with boundary conditions $\bar{w}(0) = 0$ and $\bar{w}'(\lambda) = 0$, where $\rho = \nu m$, $F_\lambda(x) = \int_0^x f(\lambda - t)dt$, and $W(x) = \int_0^x \left(\frac{2\mu}{\sigma^2} + \frac{2\rho}{\sigma^2}G_e(t)\right)dt$ with G_e being the equilibrium distribution of G.

It is well known [see, for example, Asmussen(1987, p.113)] that the unique solution of the renewal type equation in Lemma 2 is

$$\bar{w}(x) = \bar{w}'(0)(x * M)(x) - \frac{2}{\sigma^2}\int_0^x M(x-t)F_\lambda(t)dt, \tag{6}$$

where $M(x) = \sum_{n=0}^{\infty} W^{(n)}(x)$. Here, $W^{(n)}$ denotes the n-fold Stieltjes convolution of W with $W^{(0)}$ being the Heaviside function. To get $\bar{w}'(0)$, we differentiate equation (6) with respect to x and put $x = \lambda$ with boundary condition $\bar{w}'(\lambda) = 0$, then

$$\bar{w}'(0) = \frac{\frac{2}{\sigma^2}\left(\int_0^\lambda M'(\lambda-t)F_\lambda(t)dt + F_\lambda(\lambda)\right)}{M(\lambda)}.$$

Finally, $w(x) = \bar{w}(\lambda - x)$, $0 \le x \le \lambda$.

4.2 u(x), for $\tau \le x \le V$

Note again that in our model the level of water can cross λ over either through a continuous path or by jump. Hence, we first assume that V is infinite and obtain the distribution of $L(x) = Z_0(T_0) - \lambda$, the exceeding amount over λ, given that $Z_0(0) = x$, $0 \le x \le \lambda$, which is needed later to get the formula of $E[u(Z_0(T_0))|Z_0(0) = \tau]$.

Let $P_l(x) = Pr\{L(x) > l\}$, $l \ge 0$, then by an argument similar to that in Lemma 1, we have $P_l(\lambda) = 0$ and $P_l'(0) = 0$ as boundary conditions. $\bar{P}_l(x) = P_l(\lambda - x)$ satisfies the following renewal type equation:

Lemma 3.

$$\bar{P}_l(x) = \bar{P}_l'(0)x - \frac{2}{\sigma^2}\int_0^x G_l(t)dt + \int_0^x \bar{P}_l(x-t)dW(t) \tag{7}$$

with boundary conditions $\bar{P}_l(0) = 0$ and $\bar{P}_l'(\lambda) = 0$, where $G_l(x) = \rho[G_e(x+l) - G_e(l)]$.

The renewal type equation in Lemma 3 has the unique solution as follows:

$$\bar{P}_l(x) = \bar{P}_l'(0)(x * M)(x) - \frac{2}{\sigma^2}\int_0^x M(x-t)G_l(t)dt, \tag{8}$$

Differentiating the above equation with respect to x and using the boundary condition $\bar{P}_l'(\lambda) = 0$, we have

$$\bar{P}_l'(0) = \frac{\frac{2}{\sigma^2}\left(\int_0^\lambda M'(\lambda-t)G_l(t)dt + G_l(\lambda)\right)}{M(\lambda)}.$$

Finally, $P_l(x) = \bar{P}_l(\lambda - x)$. Now, when $V < \infty$, note that survival function of $L(x)$ is still $P_l(x)$, for $0 \le l < V - \lambda$, but having a discrete probability at $l = V - \lambda$, which is $P_{V-\lambda}(x)$.

Remark 1. Note that $P_0(x)$ is the probability that $Z_0(t)$ exceeds λ over by jump and $1 - P_0(x)$ is the probability that $Z_0(t)$ crosses λ over through a continuous path.

Now, we evaluate $u(x)$, for $\tau \le x \le V$. Let $\bar{u}(x) = u(V - x)$, then arguments similar to those used to derive $w(x)$ show that $\bar{u}(x)$ satisfies boundary conditions $\bar{u}(V - \tau) = 0$ and $\bar{u}'(0) = 0$, and the following renewal type equation:

Lemma 4.

$$\bar{u}(x) = \left(1 - \frac{2M}{\sigma^2}x - \frac{2\rho}{\sigma^2}\int_0^x G_e(t)dt\right)\bar{u}(0) - \frac{2}{\sigma^2}\int_0^x F_V(t)dt + \int_0^x \bar{u}(x-t)dU(t) \qquad (9)$$

with boundary conditions $\bar{u}(V-\tau) = 0$ and $\bar{u}'(0) = 0$, where $F_V(x) = \int_0^x f(V-t)dt$, and $U(x) = \int_0^x \left(\frac{2M}{\sigma^2} + \frac{2\rho}{\sigma^2}G_e(t)\right)dt$.

The unique solution of the renewal type equation in Lemma 4 is given by

$$\begin{aligned}\bar{u}(x) &= \left(N(x) - \frac{2M}{\sigma^2}(x * N)(x) - \frac{2\rho}{\sigma^2}\int_0^x N(x-t)G_e(t)dt\right)\bar{u}(0)\\ &\quad -\frac{2}{\sigma^2}\int_0^x N(x-t)F_V(t)dt, \qquad (10)\end{aligned}$$

where $N(x) = \sum_{n=0}^{\infty} U^{(n)}(x)$. To get $\bar{u}(0)$, we put $x = V - \tau$ in the above equation and use the boundary condition $\bar{u}(V-\tau) = 0$, then we have

$$\bar{u}(0) = \frac{\frac{2}{\sigma^2}\int_0^{V-\tau} N(V-\tau-t)F_V(t)dt}{\left(1 - \frac{2M(V-\tau)}{\sigma^2}\right)N(V-\tau) - \frac{2\rho}{\sigma^2}\int_0^{V-\tau} N(V-\tau-t)G_e(t)dt}.$$

Finally, $u(x) = \bar{u}(V-x)$, for $\tau \le x \le V$, and

$$E[u(Z_0(T_0))|Z_0(0) = \tau] = E[u(\lambda + L(\tau)] = \int_0^{V-\lambda} u(\lambda + l)dP_l(\tau). \qquad (11)$$

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