A Generalized MLE of the Process Change Point

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Abstract

Knowing the time of the process change could lead to quicker identification of the responsible special cause and less process down time, and it could help to reduce the probability of incorrectly identifying the special cause. In this paper, we propose a generalized maximum likelihood estimator (MLE) of the process change point when a control chart with variable sample size (VSS) scheme signals a change in the process mean, and evaluate the performance of this estimator when it is used with a VSS EWMA chart.

1. Introduction

Control charts have been widely used to monitor processes in detecting changes in the process that may result in lower-quality process output. When a control chart signals that a special cause is present, process engineers must initiate a search for and an identification of the special cause. However, the signal from a control chart does not provide process engineers with what caused the process to change or when the process change actually occurred. Knowing the time of the process change could lead to identify the special cause more quickly, and to take the appropriate actions immediately to improve quality. Consequently, estimating the time of the process change would be useful to process engineers.

The CUSUM and the EWMA charts provide built-in change point estimators from the behavior of the past plots on the control chart. Nishina(1) evaluated the performance of a CUSUM chart, an EWMA chart, and a MA chart in estimating a process change point. He concluded that the CUSUM chart is more efficient for estimating the change point than the EWMA chart and the MA chart. Samuel, Pignatiello, and Calvin(2) suggested the use of a MLE of the process change point after a Shewhart $\overline{X}$ control chart issued a signal. Pignatiello and Samuel(3) considered using the MLE of the change point instead of the built-in change point estimator when either the CUSUM or the EWMA chart issued a signal. They concluded that the performance of the MLE appears to be better than the built-in estimators over the range of magnitudes of the change considered.

In this paper, we propose a generalized MLE of the process change point when a control chart with VSS scheme signals a change in the process mean, and compare the performance of the MLE and the built-in estimator in a VSS EWMA chart. The proposed MLE is obtained by modifying the MLE proposed by Samuel, Pignatiello, and Calvin(2), to apply in cases where the sample sizes are variable.

2. Description of the VSS Control Chart

The VSS scheme allows the sample size used at each sampling point to vary depending on the previous value of the control statistic. A large sample size is used when there is some indication of a problem and a small sample size is used when there is no indication of a problem. It is shown that control charts with VSS scheme are more efficient in detecting small and moderate shifts in the process mean than control charts with the traditional fixed sample size (FSS) scheme. Control charts with VSS scheme have been studied by Prabhu, Runger, and Keats(4), Park and Reynolds(5), Costa(6), Annadi, Keats, Runger and Montgomery(7), and Zimmer, Montgomery, and Runger(8).

Consider the problem of monitoring a process and let X_t be the measurable quality characteristic of interest measured at time t $(t=1,2,\cdots)$. Assume that X_t follows a normal distribution with mean, μ, and standard deviation, σ, and the objective is to detect shifts in μ from a target value μ_0. Suppose the random samples of variable size are taken at fixed sampling intervals during the operation of the process.

Let N_t denote the sample size used at the t-th sampling time. Let $Z_t=\sqrt{N_t}(\bar{X}_t-\mu_0)/\sigma$, the standardized sample mean, and let E_t be the VSS chart statistic computed for the sample t. We assume that the VSS chart statistic, E_t, can be expressed as a function of Z_t. Then the monitoring procedure is to signal at time t if $|E_t|\geq c$ for a control limit c. If a signal is found to be a false alarm, we restart the process with $E_t=0$. The values of N_t for the sample t are determined according to the value of the previous statistic, E_{t-1}.

No optimality results are known for the choice of the sample size, but we consider only two possible sample size for simplicity. The two possible sample sizes are denoted by n_1 and n_2, where $0<n_1<n_2$. Then for $t\geq 2$, the sample size N_t can be represented as

$$N_t=\begin{cases} n_1 \text{ if } |E_{t-1}|<c_S \\ n_2 \text{ if } c_S\leq |E_{t-1}|<c \end{cases}$$

where c_S denotes the threshold limit to switch between the two sample sizes. At the start of process monitoring and after each false signal, the large sample size n_2 will be used in order to guard against problems at the start-up or possible misadjustments after repair.

The performance of a VSS control chart can be evaluated by considering the number of samples, the number of individual observations, and the time required by the chart signal. Define the average number of samples to signal (ANSS) to be expected number of samples taken from a specified starting time (frequently the start of monitoring at time $t=0$) to the time that the chart signals. Similarly, define the average number of observations to signal (ANOS) to be expected number of observations taken from a specified starting time to the time that the chart signals. Also define the average time to signal (ATS) to be the expected time from a specified starting time to the time that the chart signals. Then the average sample size is defined to be $\bar{n}=\text{ANOS}/\text{ANSS}$. For convenience we express the shift in μ units of

$$\delta=\sqrt{\bar{n}}(\mu-\mu_0)/\sigma,$$

which is the shift standardized by the standard deviation of $\bar{X}$ for the average sample size.

3. The Generalized MLE of the Process Change Point

We assume that the mean shift is occurred after an unknown point in time τ (known as the process change point), and let T denote the number of samples from the start of monitoring to the time that the chart signals. Then $\bar{X}_1,\bar{X}_2,\cdots,\bar{X}_\tau$ are the subgroup averages that are taken from the in-control process, whereas $\bar{X}_{\tau+1},\bar{X}_{\tau+2},\cdots,\bar{X}_T$ are from the changed process.

Samuel, Pignatiello, and Calvin(2) proposed a MLE of the process change point, and investigated its performance when a Shewhart $\bar{X}$ chart issues a signal. Their proposed estimators of τ, $\hat{\tau}_{MLE}$, would be

$$\hat{\tau}_{MLE}=\max_{0\leq t<T}\{t:(T-t)(\bar{Z}_{T,t})^2\}, \quad (1)$$

where $\bar{Z}_{T,t}=\sum_{i=t+1}^{T} Z_i/(T-t)$ is the overall average of the standardized sample mean for the last $T-t$ subgroups.

However, this estimator in equation (1) is not a MLE when the sample sizes are not equal. We proposed a generalized MLE of the process change point, $\hat{\tau}_{GMLE}$, which can be applied in cases where the sample sizes are variable, as follows:

$$\hat{\tau}_{GMLE}=\max_{0\leq t<T}\{t:N_{T,t}(\bar{Z}^*_{T,t})^2\}, \quad (2)$$

where $N_{T,t}=\sum_{i=t+1}^{T} N_i$ and $\bar{Z}^*_{T,t}=\sum_{i=t+1}^{T}\sqrt{N_i}Z_i/N_{T,t}$. Note that if $N_i=\bar{n}$ for all i, the generalized MLE in equation (2) reduces to the MLE in equation (1). Details about the derivation of this estimator are given in the Appendix.

4. Performance of the Generalized MLE in VSS EWMA Charts

Consider the situation to monitor the

process mean using an EWMA chart with VSS scheme. It is well known that the EWMA chart is a good alternative to the Shewhart $\overline{X}$ chart when we are interested in detecting small shifts. The VSS EWMA chart statistic is defined as

$$E_t = \lambda Z_t + (1-\lambda)E_{t-1},$$

for a weight λ $(0 < \lambda \leq 1)$ and $E_0 = 0$.

Nishina(1) proposed an built-in estimator for the process change point when an EWMA chart signals a change. If we assume $\delta > 0$ and $E_T \geq c$, the change point estimator, $\hat{\tau}_{EWMA}$, would be

$$\hat{\tau}_{EWMA} = \max_{0 \leq t < T} \{t : E_t \leq 0\}. \quad (3)$$

This estimator can be applied when the sample sizes are variable.

Now we compare the performance of the generalized MLE, $\hat{\tau}_{GMLE}$, in equation (2), with that of $\hat{\tau}_{EWMA}$ in equation (3) when a VSS EWMA chart signals a change. VSS EWMA charts will be considered with $(n_1/\overline{n}, n_2/\overline{n}) = (0.6, 2.0)$ and $(0.6, 4.0)$. It was shown that these ratio provide good statistical performance for the VSS EWMA chart (Reynolds and Arnold,(9)). We adjust λ so that the out-of-control ATS is minimized at a specified shift δ. The threshold limit, c_S, and the control limit, c, also are adjusted as λ is being adjusted so that the required in-control ATS is 370.4.

The performance of the change point estimators is given in TABLE 1 to TABLE 2 by using Monte Carlo simulation. The results of the simulation are obtained from the 100,000 simulation runs for various sizes of change in the process mean. The column labeled $E(T)$ denotes the ANSS and $E(T) - \tau$ denotes the expected number of samples taken from the process change until the signal. The column labeled $\hat{\tau}_{EWMA}$ and $\hat{\tau}_{GMLE}$ gives the average of the estimators obtained from the 100,000 simulation runs. The last column labeled $Pr(|\hat{\tau} - \tau| \leq \epsilon)$ denotes the proportion of the 100,000 runs where the estimated time of the change is within $\pm\epsilon$ of the actual change. This provides an indication of the precision of the two estimators. In the last two columns in TABLE 1 to TABLE 2, the entries in the first row are for $\hat{\tau}_{EWMA}$ and the entries in the second row are for $\hat{\tau}_{GMLE}$.

The results in TABLE 1 to TABLE 2 show that our proposed MLE, $\hat{\tau}_{GMLE}$, appears to be much less biased in estimating the process change point than $\hat{\tau}_{EWMA}$, and the precision of $\hat{\tau}_{GMLE}$ is better than that of $\hat{\tau}_{EWMA}$ for the overall range of values of δ. Specially, we note that as the amount of δ increases the proposed MLE provides much better performance than the built-in estimator.

5. Conclusions

Control charts are widely used to detect the occurrence of special causes of process shifts. When a control chart signals that a special cause is present, the signal does not provide process engineers with what caused the process to change or when the process change actually occurred. Knowing the time of the process change would help process engineers in their search for the special cause. Consequently, the estimation for the process change point can fill an important role in obtaining information for finding special causes.

Samuel, Pignatiello, and Calvin(2) proposed a MLE of the process change point after a Shewhart $\overline{X}$ chart issued a signal. Their proposed MLE can be used with CUSUM or EWMA charts when the sample sizes are all equal. However, in recent years there have been investigations of control charts in which the sampling rate is varied during the operation of the chart as a function of the data from the process. Control charts that vary the sample size are called VSS charts. It is well known that control charts with VSS scheme provide much faster detection of small and moderate shifts in the process mean than control charts with the traditional FSS scheme.

In this paper, we have proposed a generalized MLE of the process change point when a control chart with VSS scheme signals a change in the normal process mean. We also have discussed the performance of the proposed MLE when it is used with VSS EWMA charts. The results show that the proposed MLE performs well in terms of both accuracy and precision of the estimator.

Appendix: Derivation of the Generalized MLE

We assume that $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_\tau$ are the subgroup averages taken from a normal distribution, $N(\mu_0, \sigma^2)$, and

$\overline{X}_{\tau+1}, \overline{X}_{\tau+2}, \cdots, \overline{X}_T$ are from $N(\mu_1, \sigma^2)$, where μ_0 and σ are known constants and $\mu_1 = \mu_0 + \delta\sigma/\sqrt{n}$. Given the subgroup averages $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_T$, the log likelihood function can be expressed as

$$\begin{aligned} &\ln L(\tau, \mu_1 | \overline{X}_1, \cdots, \overline{X}_T) \\ &= \sum_{i=1}^{T} \ln\left(\frac{\sqrt{N_i}}{\sqrt{2\pi}}\sigma\right) - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{\tau} N_i(\overline{X}_i - \mu_0)^2 \right. \\ &\left. + \sum_{i=\tau+1}^{T} N_i(\overline{X}_i - \mu_1)^2\right] \\ &= \sum_{i=1}^{T} \ln\left(\frac{\sqrt{N_i}}{\sqrt{2\pi}\sigma}\right) - \frac{1}{2}\left[\sum_{i=1}^{T} Z_i^2 \right. \\ &- 2(\mu_1 - \mu_0)\sum_{i=\tau+1}^{T} \frac{\sqrt{N_i}Z_i}{\sigma} \\ &\left. + (\mu_1 - \mu_0)^2 \sum_{i=\tau+1}^{T} \frac{N_i}{\sigma^2}\right]. \end{aligned}$$

If the change point τ were known, the MLE of μ_1, $\widehat{\mu_1}$, would be

$$\widehat{\mu_1} = \mu_0 + \sigma \overline{Z}^*_{T,\tau}.$$

Substituting this estimator into the log likelihood function, we get

$$\begin{aligned} &\ln L(\tau | \overline{X}_1, \cdots, \overline{X}_T) \\ &= \sum_{i=1}^{T} \ln\left(\frac{\sqrt{N_i}}{\sqrt{2\pi}\sigma}\right) - \frac{1}{2}\sum_{i=1}^{T} Z_i^2 + \frac{1}{2} N_{T,\tau}(\overline{Z}^*_{T,\tau})^2 \end{aligned}$$

Therefore the value of τ that maximizes the log likelihood function is

$$\hat{\tau}_{GMLE} = \max_{0 \le t < T} \{ t : N_{T,t}(\overline{Z}^*_{T,t})^2 \}.$$

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TABLE 1. Change point estimators and their precision in VSS EWMA charts ($\tau = 100$ and λ is optimal for $\delta = 0.5$)

$n_1/\bar{n}$ $n_2/\bar{n}$	λ	c_S c	δ	$E(T)$	$\hat{\tau}_{EWMA}$ $\hat{\tau}_{CMLE}$	$Pr(\lvert\hat{\tau}-\tau\rvert \le \epsilon)$ ϵ = 0	1	2	3
0.6 2.0	0.091	0.223 0.584	0.5	120.06	101.39 103.81	0.06 0.08	0.16 0.18	0.24 0.26	0.31 0.32
			0.75	111.51	97.26 99.82	0.10 0.16	0.24 0.32	0.35 0.43	0.45 0.52
			1.0	108.10	95.97 99.14	0.13 0.25	0.31 0.45	0.44 0.58	0.54 0.67
			1.5	105.21	95.04 99.24	0.21 0.43	0.43 0.66	0.56 0.78	0.63 0.85
			2.0	103.90	94.60 99.46	0.28 0.57	0.50 0.80	0.60 0.89	0.65 0.93
			3.0	102.67	94.43 99.79	0.41 0.78	0.56 0.94	0.62 0.97	0.66 0.99
0.6 4.0	0.138	0.410 0.757	0.5	118.37	104.50 104.07	0.06 0.08	0.17 0.18	0.25 0.26	0.33 0.33
			0.75	110.41	99.25 100.44	0.11 0.16	0.26 0.32	0.39 0.43	0.49 0.52
			1.0	107.36	97.61 99.76	0.16 0.24	0.35 0.45	0.49 0.58	0.59 0.67
			1.5	104.75	96.56 99.69	0.25 0.41	0.48 0.65	0.60 0.78	0.67 0.85
			2.0	103.62	96.22 99.80	0.33 0.54	0.54 0.79	0.63 0.89	0.69 0.94
			3.0	102.68	95.98 99.97	0.43 0.76	0.58 0.93	0.64 0.98	0.69 0.99

TABLE 2. Change point estimators and their precision in VSS EWMA charts ($\tau = 100$ and λ is optimal for $\delta = 1.0$)

$n_1/\bar{n}$ $n_2/\bar{n}$	λ	c_S c	δ	$E(T)$	$\hat{\tau}_{EWMA}$ $\hat{\tau}_{GMLE}$	$Pr(\lvert\hat{\tau}-\tau\rvert \le \epsilon)$ $\epsilon = 0$	1	2	3
0.6 2.0	0.246	0.393 1.084	0.5	125.68	112.11 105.73	0.07 0.08	0.17 0.17	0.25 0.25	0.32 0.31
			0.75	111.39	100.71 100.62	0.15 0.16	0.34 0.32	0.47 0.44	0.57 0.52
			1.0	107.05	98.76 99.40	0.22 0.26	0.44 0.47	0.58 0.60	0.68 0.69
			1.5	104.11	97.83 99.28	0.32 0.43	0.55 0.68	0.67 0.81	0.74 0.88
			2.0	102.98	97.57 99.50	0.40 0.57	0.60 0.81	0.69 0.91	0.75 0.95
			3.0	102.09	97.42 99.81	0.48 0.78	0.61 0.94	0.69 0.98	0.75 0.99
0.6 4.0	0.353	0.713 1.364	0.5	124.34	114.59 105.77	0.07 0.08	0.16 0.17	0.23 0.25	0.30 0.31
			0.75	110.33	102.24 100.83	0.15 0.16	0.33 0.32	0.46 0.44	0.56 0.53
			1.0	106.45	99.76 99.78	0.23 0.25	0.45 0.46	0.60 0.60	0.70 0.69
			1.5	103.86	98.49 99.63	0.34 0.41	0.58 0.66	0.70 0.80	0.78 0.88
			2.0	102.91	98.18 99.82	0.42 0.55	0.62 0.80	0,73 0.91	0.80 0.95
			3.0	102.12	98.03 99.96	0.48 0.76	0.64 0.94	0.73 0.99	0.80 1.00