Structural results and a solution for the product rate variation problem : A graph-theoretic approach

Choe, Sang-Woong

Pohang College, Korea

swchoe@pohang.ac.kr

Abstract

The product rate variation problem, to be called the PRVP, is to sequence different type units that minimizes the maximum value of a deviation function between ideal and actual rates. The PRVP is an important scheduling problem that arises on mixed-model assembly lines. A surge of research has examined very interesting methods for the PRVP. We believe, however, that several issues are still open with respect to this problem. In this study, we consider convex bipartite graphs, perfect matchings, permanents and balanced sequences. The ultimate objective of this study is to show that we can provide a more efficient and in-depth procedure with a graph theoretic approach in order to solve the PRVP. To achieve this goal, we propose formal alternative proofs for some of the results stated in the previous studies, and establish several new results.

Keyword : Convex bipartite graph, Perfect matching, Permanent, Balanced sequence

1 Introduction

The product rate variation problem, to be called the PRVP, is to sequence different product units that minimizes the maximum value of a deviation function between ideal and actual rates. The PRVP is an important scheduling problem that arises on mixed-model assembly lines. And it is possible for us to exhibit the link between balanced sequences (more precisely, word combinatorics) and this scheduling problem.

The PRVP is given by a number n of different *products* and a *required demand* d_i for each product $i\,(i = 1, 2, \ldots, n)$. Then, we may denote a *total required demand* by $D = \sum_{i=1}^{n} d_i$ and an *ideal rate* for each product i by $r_i = \frac{d_i}{D}$. The term "ideal" refers to the fact that we would like to allocate each product i in a sequence in proportion r_i. Such a sequence would be *balanced, uniformly leveled* and *fair.* In addition, we let $x_{it}\,(t = 1, 2, \ldots, D)$ be an *actual demand* of each product i up to time (or position) t of a given sequence. Thus, $x_{it} = j\,(j = 1, 2, \ldots, d_i)$ if the number of appearances of the product i up to time t is exactly j. And tr_i means an *ideal demand* for the product i up to time t.

In approaching the PRVP, two basic considerations usually arise. The one involves determining a standard that constitute an ideal fair allocation. In other words, to define a deviation function between the ideal and actual demands is a key issue. There are many possible definitions for a deviation function. A good standard is the one which is accepted by the researchers involved. The other is to construct other objectives and constraints which enable us to obtain an *optimal sequence.* As a matter of course, we can not ensure that the PRVP is a *unimodal problem* so that an optimal sequence is unique under any situation.

The roots of the general approach we employ lie in the study that has been analyzed in [5]. The authors gave a very interesting analysis of the PRVP. We believe, however, that several issues are still open with respect to this problem.

The ultimate objective of this study is to show that we can provide a more efficient and in-depth procedure with a graph theoretic approach, not an integer programming one, in order to solve the PRVP. To achieve this goal, we propose formal alternative proofs for some of the results stated in [5], and establish several new results and problems in terms of graph theory.

In section 2, we state one of the problems related to the PRVP. Then we consider several theories with respect to convex bipartite graphs and a matching problem in order to prove structural results. Section 3 contains more precise descriptions of the PRVP and related issues. Section 4 provides proofs of structural results and several examples. Finally, in section 5, we state a summary, relations to preceding results and further studies.

2 Preliminaries

2.1 A problem related to the PRVP

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^{\mathbb{Z}}$ the set of sequences defined on $\mathcal{A}$. For $u := u_1u_2u_3\cdots \in \mathcal{A}^{\mathbb{Z}}$, a *word* w of u is a finite subsequence of consecutive letters in u: for example, $w = u_hu_{h+1}...u_{h+k-1}$. The integer k is the *length* of w and will be denoted by $|w|$. For $a \in \mathcal{A}$, let $|w|_a$ denote the number of a's in the word w.

Definition 1. ([3]) The sequence $u \in \mathcal{A}^{\mathbb{Z}}$ is *balanced* if for any two words w and w' in u of the same length and any $a \in \mathcal{A}$, $-1 \leqslant |w|_a - |w'|_a \leqslant 1$.

Definition 2. If an n-tuple $(p_1, p_2, \ldots, p_n)$ of rates such that $\sum_i p_i = 1$ makes a balanced sequence, the n-tuple is said to be *balanceable.*

Theorem 1 ([3, Theorem 2.18]). *For any sequence composed of two types, the set of rates $(p, 1-p)$ is balanceable.*

Theorem 2 ([3, Theorem 2.19]). *For any sequence composed of three types, the set of rates (p_1, p_2, p_3) is balanceable if and only if $(p_1, p_2, p_3) = (4/7, 2/7, 1/7)$ or two rates are equal.*

Theorem 3 ([3, Theorem 2.20]). *For any sequence composed of four types, the rate tuple (p_1, p_2, p_3, p_4) with four distinct rates is balanceable if and only if $(p_1, p_2, p_3, p_4) = (8/15, 4/15, 2/15, 1/15)$.*

Proposition 1 ([3, Proposition 2.21]). *For any sequence composed of four types, if the rate tuple (p_1, p_2, p_3, p_4) is made of less than two distinct numbers, then it is balanceable.*

Lemma 1 ([3, Lemma 2.26]). *Let w be balanced with rates $p_1 > \ldots > p_n$, then w is periodic. In particular, $p_i \in \mathbb{Q}$ for all i.*

2.2 Convex bipartite graphs

We introduce some definitions and results for bipartite graphs. For the terminology and notation not defined in our discussion, we refer to [17, 14]. A bipartite graph $G = (V_r, V_c, E)$ is an undirected graph with a vertex set $V_r \cup V_c$ and an edge set $E \subseteq (V_r \times V_c)$, where V_r and V_c are disjoint.

Definition 4. A *convex bipartite graph* is a bipartite graph $G = (V_r, V_c, E)$ with an ordering $V_c := (v_{c1}, v_{c2}, \ldots, v_{cn})$ such that, for any $v_r \in V_r$, if $(v_r, v_{ci}) \in E$ and $(v_r, v_{cj}) \in E$ then $(v_r, v_{ck}) \in E$ for all $i \leq k \leq j$.

A convex bipartite graph was originally discussed by Glover ([4]). It is a bipartite graph $G = (V_r, V_c, E)$ given by specifying an ordering "$\leq$" and by specifying the endpoints and the interval of the elements of V_c connected to v_r for every $v_r \in V_r$.

An example of a convex bipartite graph is given on Figure 1. Here, a vertex (i, j) adjacent to a vertex t indicates $x_{it} = j$ in the PRVP described in the pervious section. Observe that this instance admits a perfect matching indicated as thick solid lines. Thus there exists an optimal sequence, say $(1, 2, 3, 1, 2)$.

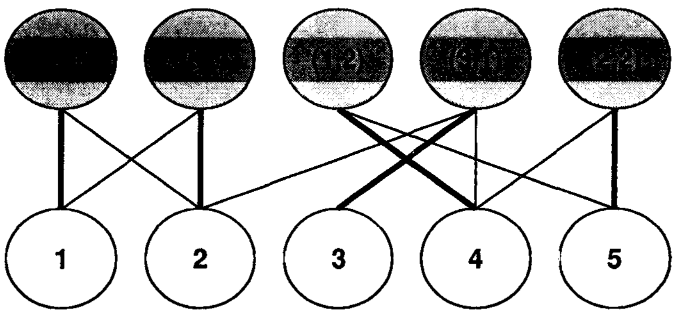

Figure 1: Convex bipartite graph with $(d_1, d_2, d_3) = (2, 2, 1)$

2.3 Perfect matchings

2.3.1 General results

A *matching* M in a graph $G = (V, E)$ is a subset of E such that no two edges in M are incident to the same vertex. *Maximum matching* in a graph $G = (V, E)$ is a matching of G of which the size (number of edges) is maximum.

A subset $V_s \subseteq V$ is called a *vertex cover* of G if and only if every edge of G is incident to at least one vertex in V_s. The minimum size of vertex covers of G is called the *vertex covering number* of G. An *edge cover* of G is a set of edges such that every vertex in V is incident to an edge in the set. The minimum size of edge covers of G is called the *edge covering number* of G.

A set of vertices is *independent* if there is no edge between any two of them. The size of any maximum independent set is called the *independent number* of G.

Now, we shall show some general results with respect to matchings. To begin with, we shall use some notations as follows:

- The matching number of G is denoted by $\nu(G)$;
- The vertex covering number of G is denoted by $\tau(G)$;
- The edge covering number of G is denoted by $\rho(G)$;

- The independent number of G is denoted by $\alpha(G)$;
- The set of neighbors of $V_s \subseteq V$ in a graph $G = (V, E)$ is denoted by $N(V_s)$.

Theorem 4 (Gallai(1959)). *For any graph $G = (V, E)$, let $n := |V|$, then*

$$(i)\ \alpha(G) + \tau(G) = n$$
$$(ii)\ \nu(G) + \rho(G) = n \tag{1}$$

if G has no isolated vertex.

Hereinafter, we restrict our attention to bipartite graphs only.

Definition 5 . In a bipartite graph $G = (V_r, V_c, E)$, a *complete matching* from V_r to V_c is a matching M such that every vertex in V_r is incident to an edge in M, and a *perfect matching* is a matching that is complete from V_r to V_c as well as from V_c to V_r.

If both V_r and V_c have the same number of vertices, a complete matching from one to the other is a perfect matching. In other words, a perfect matching in a bipartite graph $G = (V_r, V_c, E)$ defines an injective mapping $f : V_r \longrightarrow V_c$ such that for every $v_r \in V_r$, there is an edge $e := (v_r, f(v_r)) \in E$.

Theorem 5 (König(1931)). *If $G = (V, E)$ is a bipartite graph, then*

$$\tau(G) = \nu(G). \tag{2}$$

This theorem is also referred to as the König-Egerváry theorem as Egerváry came up with the same result in [8].

When does a bipartite graph have a complete matching? Given a graph, if we wish to prove that the graph has a complete matching, we can simply give the edges in the matching. On the other hand, how do we prove that a graph has no complete matching? We state Hall's theorem which gives a necessary and sufficient condition for the existence of a complete matching in a bipartite graph.

Given a bipartite graph $G = (V_r, V_c, E)$ and a subset of vertices $U \subseteq V_r$, the neighborhood $N(U)$ is the subset of vertices of V_c that are adjacent to some vertex in U, i.e.

$$N(U) = \{v_c \in V_c \mid (u, v_c) \in E \text{ for some } u \in U\} = \bigcup_{u \in U} N(u). \tag{3}$$

Theorem 6 (Hall(1935)). *Let $G = (V_r, V_c, E)$ be a bipartite graph with $|V_r| \leq |V_c|$. Then $G = (V_r, V_c, E)$ has a complete matching from V_r to V_c if and only if $|N(U)| \geq |U|$ for all $U \subseteq V_r$.*

Theorem 7 (Frobenius(1917)). *Let $G = (V_r, V_c, E)$ be a bipartite graph. Then $G = (V_r, V_c, E)$ has a perfect matching*[†] *from V_r to V_c if and only if $|V_r| = |V_c|$ and $|N(U)| \geq |U|$ for all $U \subseteq V_r$.*

Frobenius theorem is often called the marriage theorem. It is interesting to note that all three theorems are equivalent.

Theorem 8 (Equivalence of König, Frobenius and Hall's theorems). *There exists a circular implication such that König $\Rightarrow$ Hall, Hall $\Rightarrow$ Frobenius and Frobenius $\Rightarrow$ König.*

2.3.2 Permanents

Definition 8 . Let $G = (V, E)$ be a bipartite graph with bipartition (V_r, V_c). The $|V_r|$ by $|V_c|$ matrix $\tilde{\mathbf{X}}$, defined by $\tilde{X}_{ij} = 1$ if $(i, j) = e \in E$ and $\tilde{X}_{ij} = 0$ otherwise, is called $\tilde{\mathbf{X}}$ the *bipartite adjacency matrix* of a graph G.

Definition 9 . The *adjacency matrix* of a graph $G = (V, E)$ is the $|V| \times |V|$ matrix $\mathbf{A}$ whose elements A_{ij} are given such that $A_{ij} = 1$ if v_i and v_j are adjacent, and $A_{ij} = 0$ otherwise.

Is there any method so that we may verify the existence of a perfect matching by means of a bipartite adjacency matrix or an adjacency matrix? The answer is "yes". In this subsection, we shall introduce the permanent which enables us to check the number of perfect matchings.

The computation of a permanent has been studied extensively in algebraic complexity theory, which is known as a ♯P-complete problem ([11]).

Definition 10 . Let $\mathbf{C} = (c_{ij})$ be a square matrix of order n over a ring R. The *permanent* of $\mathbf{C}$ is defined by

$$\mathrm{per}(\mathbf{C}) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} c_{i\,\sigma(i)}, \tag{4}$$

where S_n denotes the symmetric group of degree n.

Let $\mathbf{C}_{(i,j)}$ be the matrix obtained from a square matrix $\mathbf{C} = (c_{ij})$ by deleting the $i-th$ row and the $j-th$ column. Then it is also easy to see that for any i and j,

$$\mathrm{per}(\mathbf{C}) = \sum_{k=1}^{n} \left(c_{ik}\,\mathrm{per}(\mathbf{C}_{(i,k)})\right) = \sum_{k=1}^{n} \left(c_{kj}\,\mathrm{per}(\mathbf{C}_{(k,j)})\right). \tag{5}$$

[†]A perfect matching is also referred as *marriage or assignment.*

Theorem 9 ([6, Minc(1978)]). *Let $G = (V, E)$ be a bipartite graph with bipartition (V_r, V_c) such that $|V_r| = |V_c|$. And let* $\mathbf{A}$ *and* $\tilde{\mathbf{X}}$ *be the adjacency matrix and the bipartite adjacency matrix of the graph $G = (V, E)$, respectively. Namely,*

$$\mathbf{A}_{|V|\times|V|} = \begin{matrix} & V_r & V_c \\ V_r & \mathbf{O} & \tilde{\mathbf{X}} \\ V_c & \tilde{\mathbf{X}}^{\mathrm{t}} & \mathbf{O} \end{matrix} .$$

Then the number of perfect matchings of the graph $G = (V, E)$ is given by

$$|\mathcal{M}| = \sqrt{\mathrm{per}(\mathbf{A})} = \mathrm{per}(\tilde{\mathbf{X}}). \tag{6}$$

For example, consider the bipartite adjacency matrix and the adjacency matrix of the graph in Figure 1, where $V_r = \{(1,1),(2,1),(1,2),(3,1),(2,2)\}$, $V_c = \{1,2,3,4,5\}$ and $V = V_r \bigcup V_c$.

Note that,

$$\begin{aligned} \mathrm{per}(\tilde{\mathbf{X}}) &= \sum_{\sigma \in S_5} \prod_{i=1}^{5} \tilde{X}_{i\,\sigma(i)} \\ &= \tilde{X}_{43} \cdot \tilde{X}_{11} \cdot \tilde{X}_{22} \cdot \tilde{X}_{34} \cdot \tilde{X}_{55} + \tilde{X}_{43} \cdot \tilde{X}_{11} \cdot \tilde{X}_{22} \cdot \tilde{X}_{54} \cdot \tilde{X}_{35} \\ &\quad + \tilde{X}_{43} \cdot \tilde{X}_{12} \cdot \tilde{X}_{21} \cdot \tilde{X}_{34} \cdot \tilde{X}_{55} + \tilde{X}_{43} \cdot \tilde{X}_{12} \cdot \tilde{X}_{21} \cdot \tilde{X}_{54} \cdot \tilde{X}_{35} = 4\,, \\ \mathrm{per}(\mathbf{A}) &= \left[\mathrm{per}(\tilde{\mathbf{X}})\right]^2 = 16\,. \end{aligned}$$

Besides, we see that $\mathrm{per}(\tilde{\mathbf{X}})$ and $\mathrm{per}(\mathbf{A})$ enumerate the four perfect matchings. In this instance, there exist four optimal sequences

$$(1,2,3,1,2),\ (1,2,3,2,1),\ (2,1,3,1,2),\ (2,1,3,2,1),$$

respectively. Thus we have reformulated the bipartite matching problem as an algebraic complexity problem via Minc's theorems. Minc ([6]) has showed that the permanent of a bipartite adjacency matrix enumerates all the perfect matchings of a graph $G = (V_r, V_c, E)$. In other words, if $\mathrm{per}(\tilde{\mathbf{X}}) \neq 0$, then by considering the permutation expansion of the permanent we obtain

$$\mathrm{per}(\tilde{\mathbf{X}}) = \sum_{m \in \mathcal{M}} \prod_{e:=(i,j)\in m} \tilde{X}_e\,, \tag{7}$$

where $\mathcal{M}$ is the set of all the perfect matchings in the graph $G = (V_r, V_c, E)$. Therefore, the permanent provides an useful information of the existence and the numbers of a perfect matching. In short, the relation between the perfect matching and the permanent in a graph $G = (V_r, V_c, E)$ such that $|V_r| = |V_c|$ is given on Figure 2.

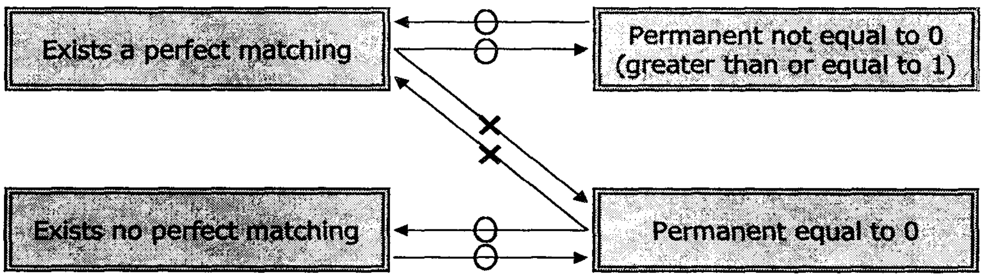

Figure 2: Relation between the perfect matching and the permanent in a graph $G = (V_r, V_c, E)$ such that $|V_r| = |V_c|$

2.4 Colorings of graphs

A graph is said to be *k-colorable* if it is possible to assign one color from a set of k colors to each vertex such that no two adjacent vertices have the same color.

Definition 11 . If a graph is k-colorable but not $(k-1)$-colorable, we say that the graph is *k-chromatic* and that its *chromatic number* is k.

So the chromatic number is the minimum number k such that a graph is k-colorable. Hence a graph is k-colorable if and only if its chromatic number is less than or equal tc k. In other words, a k-chromatic graph is a graph that needs at least k colors, whereas a k-colorable graph is a graph that does not need more than k colors. Obviously, a trivial graph is the chromatic number of 1 and a bipartite graph is the chromatic number of 2.

A simple graph is said to be *k edge colorable* if it is possible to assign one color from a set of k colors to each edge such that no two edges with a vertex in common get the same color.

Definition 12 . If a graph is k edge colorable but not $(k-1)$ edge colorable, we say that the graph is *k edge chromatic* and that its *chromatic index* is k.

So the chromatic index is the minimum number k such that a graph is k edge colorable. Obviously, the maximum degree of any graph is necessarily a lower bound for its chromatic index, whereas by Brooks's theorem, the maximum degree is an upper bound of the chromatic number of any graph that is neither a complete graph nor an odd cycle ([17, page 255]).

Definition 13 . A k-chromatic graph $G = (V, E)$ is *uniquely colorable* if any k-coloring of G induces the same partition of the vertex set of G.

Some notations are given as follows:

- The chromatic number of a graph $G = (V, E)$ is denoted by $\chi(G)$;
- The chromatic index of a graph $G = (V, E)$ is denoted by $\chi_I(G)$;
- The maximum degree of a graph $G = (V, E)$ is denoted by $\Delta(G)$.
- The minimum degree of a graph $G = (V, E)$ is denoted by $\delta(G)$.

Theorem 10 (König's line coloring theorem(1916)). *For every bipartite graph* $G = (V_r, V_c)$, $\chi_I(G) = \Delta(G)$.

Proposition 2 ([17, page 260]). *The only uniquely colorable 2-chromatic graphs are the bipartite graphs.*

Proposition 3 ([17, pages 257, 260]). *If the k-chromatic graph* $G = (V, E)$ *is uniquely colorable,* $\delta(G) \geq (k - 1)$.

3 The minimization problem of a product rate variation

In this section, we shall review the definition and the goal of the PRVP, limitations of Integer Programming Approach and benefits of Graph Theoretic Approach. And then, we shall describe our problem.

We shall use the following notations:

- $\lfloor a \rfloor$ will denote the largest integer smaller or equal to a for any real number a;
- $\lceil a \rceil$ will denote the smallest integer larger or equal to a for any real number a;
- $\mathcal{N}$ is the set of all nonnegative integers;

3.1 Literature review

In the formulation of the PRVP, a key issue is how the deviation function between ideal and actual demands is defined. The Minkowski's L_p metric defines distance between two points. Usually, the L_p metric is especially operationally important when $p = 1, 2, \infty$. L_1 (Manhattan distance) and L_2 (Euclidean distance) are the longest and the shortest distances in the geometrical sence. L_∞ (Tchebycheff distance) is the shortest distance in the numerical sence. In this study, fortunately, because of commensurability between ideal and actual demands, it is possible to directly use any distance of the above three metrics. This implies that it is not necessary for us to normalize the ideal demand $(t\,r_i)$ and the actual demand (x_{it}). On the other hand, Balinski et al. ([12]) have concluded that an arbitrary choice of a distance function is misdirected, and have recommended a normalization.

In this paper, we adopt L_∞ (Tchebycheff distance) as the definition of deviation between ideal and actual demands. Then, for each t $(t = 1, 2, \ldots, D)$, a deviation between the ideal and the actual demands is given by

$$L_\infty(t) \;=\; \| \mathbf{x}_t - \mathbf{r}_t \|_\infty \;=\; \max_i | x_{it} - tr_i |, \tag{8}$$

where $\mathbf{x}_t := (x_{1t}, x_{2t}, \ldots, x_{nt})^t$ and $\mathbf{r}_t := (tr_1, tr_2, \ldots, tr_n)^t$. Thus, a maximum deviation is also defined by

$$\max_t L_\infty(t) = \max_t \| \mathbf{x}_t - \mathbf{r} \|_{\infty,t} = \max_t(\max_i | x_{it} - tr_i |) = \max_{i,t} | x_{it} - tr_i |. \tag{9}$$

We can formulate the PRVP as an optimization problem as follows ([5, 12]).

[P1] **Minimize.**

$$\max_{i,t} | x_{it} - tr_i | \tag{10}$$

Subject to.

$$\sum_{i=1}^{n} x_{it} = t, \qquad t = 1, 2, \ldots, D. \tag{11}$$

$$x_{iD} = d_i, \qquad i = 1, 2, \ldots, n. \tag{12}$$

$$x_{i(t+1)} \geq x_{it}, \qquad i = 1, 2, \ldots, n,\; t = 1, 2, \ldots, D-1. \tag{13}$$

$$x_{it} \in \mathcal{N}, \qquad i = 1, 2, \ldots, n,\; t = 1, 2, \ldots, D. \tag{14}$$

In this formulation, Equation (11) means that t objects have to be allocated during the first t positions, and Equation (12) describes the required frequency constraints. Inequality (13) indicates an actual frequency should not decrease with time. The aforementioned formulation shows that given n rational numbers $r_1, r_2, \ldots, r_n$ with a common denominator D, the PRVP is to find nD integers x_{it} which construct an optimal sequence under the restrictions (11)-(14).

On the other hand, it is worth while for us to consider the following PRVP **[P2]** equivalent to **[P1]**. In fact, **[P2]** may be more efficient than **[P1]** in manipulating the problem.

[P2] **Minimize.**

$$\beta := \max_{i,t} | x_{it} - tr_i | \tag{15}$$

Subject to.

$$| x_{it} - tr_i | \leq \beta, \qquad i = 1, 2, \ldots, n,\; t = 1, 2, \ldots, D. \tag{16}$$

$$\sum_{i=1}^{n} x_{it} = t, \qquad t = 1, 2, \ldots, D. \tag{17}$$

$$x_{iD} = d_i, \qquad i = 1, 2, \ldots, n. \tag{18}$$

$$x_{i(t+1)} \geq x_{it}, \qquad i = 1, 2, \ldots, n,\; t = 1, 2, \ldots, D-1. \tag{19}$$

$$x_{it} \in \mathcal{N}, \qquad i = 1, 2, \ldots, n,\; t = 1, 2, \ldots, D. \tag{20}$$

The difference between **[P1]** and **[P2]** is two-fold. The one is to convert the objective function (11) into (16) and the other is to add the constraint (16) to **[P2]**. In general, there exist three approaches to solve the PRVP. They are
(1) Bottleneck Assignment Problem ([7, 2]),
(2) Sequence of Assignment Problem and
(3) Sequence of Matching Problem ([5]).
The third approach involves an application of a bipartite maximum cardinality matching algorithm.

Both **[P1]** and **[P2]** are integer programming problems such as a bottleneck assignment problem and a sequence of assignment problem. Hereinafter, we shall focus our attention on **[P2]**. Now, we define a set $\mathcal{S}$ of all optimal sequences in **[P2]** as follows:

$$\mathcal{S} := \{\mathbf{s} \mid \mathbf{s} = (s_1, s_2, \dots, s_D), \mid x_{s_t t} - tr_{s_t}| \leq \beta^*, x_{s_t t} > x_{s_t (t-1)}, t = 1, 2, \dots, D.\}, \quad (21)$$

where $\beta^* := \min \beta$. The goal of **[P2]** may be described as follows:

- Input n and d_i for each i and find x_{it} values such that $\mid x_{it} - tr_i \mid \ \leq \ \beta^*$, for all i and t.
- And then, determine $\mathcal{S}$ with the values of x_{it}.

If we approach **[P2]** with a numeric analysis such as a Branch and Bound Method to the Integer Programming, only one optimal sequence may be computed. That is, this approach always finds only one sequence in $\mathcal{S}$. This is the reason why we apply the graph theoretic methodology to PRVP **[P2]**. Elegant theoretic findings such as the expansion of the permanent of a bipartite adjacency matrix ([6]) and the perfect matching algorithm ([9, 10]) leads us to the true cardinality of $\mathcal{S}$. It will become clear below that the problem of finding all optimal solutions (or sequences) for the PRVP **[P2]** can be reduced to the problem of catching all perfect matchings in a certain convex bipartite graph.

We now briefly summarize some of main results concerning the PRVP.

- Steiner et al. ([5]) have introduced a maximum deviation problem with L_∞. They have showed that a sequence always exists such that the deviation of an actual demand from the ideal demand of all products is never greater than one. The authors have developed an optimization procedure for the PRVP considering it as a matching problem in a bipartite graph and have demonstrated that β^* belongs to the interval $[1 - r_{\max}, 1)$, where $r_{\max} := \max_i r_i$.
- Balinski et al. ([12]) have also noted the connection between the PRVP and apportionment problems. From this connection, and from known results concerning apportionment problems, it is not too difficult to deduce that simple-minded procedure such as [5] do not provide an optimal solution for the PRVP in terms of a total deviation, denoted by $\sum_{i,t} |x_{it} - tr_i|$.

- Altman et al. ([3]) have exhibited the link between balanced sequences and the routing problem in queueing networks. This study is essentially based on the word combinatorics and uses its specific vocabulary, which relies heavily on results in [15, 13, 16, 1].
- Minc ([6]) has showed very elegant results which describe the existence and the numbers of a perfect matching in a bipartite graph. He has stated that the expansion of the permanent of a bipartite adjacency matrix enables us to get all the perfect matchings.
- Fukuda et al. ([9, 10]) have proposed an algorithm which finds all the perfect matchings in an unweighted or a weighted bipartite graph. Their algorithm is as follows:
  - First, they check whether or not at least one perfect matching exists by solving the maximum cardinality matching problem.
  - If a perfect matching exists, then they check again if there exists a different perfect matching. If a different perfect matching exists, they show that all the perfect matchings can be obtained by generating a sequence of these subproblems iteratively.

To have a proper understanding of those things mentioned in the previous paragraphs, let us illustrate an example.

Example 1. Consider the instance $n = 3, d_1 = 3, d_2 = 3, d_3 = 1$. Table 1 gives an optimal solution for x_{it}'s and β which are numerically analyzed by the method of the integer programming.

Table 1: IP approach : x_{it}'s optimal values under $\mathbf{d} = (3,3,1)$ and $\beta^* = \frac{5}{7}$

$i \backslash t$	1	2	3	4	5	6	7
1	0	1	2	2	2	2	3
2	1	1	1	2	2	2	3
3	0	0	0	0	1	1	1

Table 1 shows that an optimal sequence is $\mathbf{s} = (s_1, s_2, \ldots, s_7) = (2, 1, 1, 2, 3, 2, 1)$, since $x_{21} > x_{20}$, $x_{12} > x_{11}$, $x_{13} > x_{12}$, $x_{24} > x_{23}$, $x_{35} > x_{34}$, $x_{26} > x_{25}$, $x_{17} > x_{16}$. As mentioned previously, the integer programming approach provides only one optimal sequence in any case, which is the limitation of a numeric analysis. More preciously, all nonlinear optimization techniques assume the unimodal problem and the compactness.

In case of infinite sequences, the required frequency $(d_1, d_2, d_3) = (3, 3, 1)$ is balanceable ([3, 16]). In this paper, however, we consider only finite sequences. Thus, the number of optimal sequences, denoted by $|S|$, may be greater than one in a certain situation. Actually, there exist 24 optimal sequences ($|S| = 24$) in Example 1 by means of the expansion of the determinant of a bipartite matching matrix or the expansion of the permanent of a bipartite adjacency matrix. These facts imply that we need to change our direction from the integer programming approach to the graph theoretic one.

3.2 Statement of problems

The ultimate objective of this study is to show that we could provide a more efficient and more in-depth procedure with a graph theoretic approach, not an integer programming one, in order to solve the PRVP.

To begin with, we shall use some notations as follows:

- $G(\mathbf{d},\beta^*) := (\mathcal{V},\mathcal{W},\mathcal{E} \mid \mathbf{d},\beta^*)$ will denote a bipartite graph corresponding to the PRVP **[P2]** which is given by a pair $(\mathbf{d},D)$ where $\mathbf{d}=(d_1,d_2,\ldots,d_n)$ and $D=\sum_i d_i$;
- $\mathcal{V}$ and $\mathcal{W}$ are disjoint vertex sets with the same cardinality such that $\mathcal{V} := \{v_{ij} \mid i = 1,2,\ldots,n,\ j=1,2,\ldots,d_i.\}$ and $\mathcal{W} := \{w_t \mid t=1,2,\ldots,D.\}$;
- $\mathcal{E}$ is an edge set such that $\mathcal{E} := \{e_{ij,t} := (v_{ij},w_t) \mid x_{it}=j\} \subseteq (\mathcal{V}\times\mathcal{W})$;
- $\mathcal{M}$ will denote the set of all the perfect matchings in the graph $G(\mathbf{d},\beta^*)$.

Steiner et al. ([5]) have proposed the following inequality for the PRVP **[P2]**.

$$0 \leqslant \frac{j-\beta}{r_i}-1 \leqslant t \leqslant \frac{j-1+\beta}{r_i} \leqslant D-1. \tag{22}$$

In view of this paper, it is equivalent to

$$1 \leqslant \frac{j-\beta}{r_i} \leqslant t \leqslant \frac{j-1+\beta}{r_i}+1 \leqslant D. \tag{23}$$

Thus, their result means that an integer t lies in the interval $\Big[\lceil\frac{j-\beta}{r_i}\rceil, \lfloor\frac{j-1+\beta}{r_i}+1\rfloor\Big]$. Also, they have stated that the value β belongs to the interval $[1-r_{\max},1)^\dagger$. Unfortunately, they have passed through an obscure and informal process of derivation.

The objectives pursued in this study are as follows:

(1) We propose two formal alternative proofs for the results stated in [5]: the lower bound of β^* and the degree of vertex v_{ij}.

(2) Also, we provide two new results not stated in [5, 12]: the degree of vertex w_t and the refined lower bound of β^*.

(3) Based on (1) and (2) above, we introduce a procedure for the construction of the graph $G(\mathbf{d},\beta^*)$.

(4) We prove that the graph $G(\mathbf{d},\beta^*)$ is always convex with respect to the vertex set $\mathcal{W}$.

(5) We confirm that an optimal sequence corresponds to a perfect matching in the graph

†see their Lemma 1,2,3 and Theorem 1,2.

$G(\mathbf{d}, \beta^*)$. That is, $|\mathcal{S}| = |\mathcal{M}|$ and there exists a bijection from $\mathcal{S}$ to $\mathcal{M}$. This implies that an optimal sequence in a integer programming approach is equivalent to a perfect matching in a graph theoretic approach.

4 Structural results for the PRVP

4.1 Bounds on the value β

Let us start with defining some notations:

- $\deg(v_{ij})$ and $\deg(w_t)$ are degrees of v_{ij} and w_t, respectively;
- We shall denote a subset $\mathcal{P}_i$ of the vertex set $\mathcal{V}$ as $\mathcal{P}_i := \{v_{i1}, v_{i2}, \ldots, v_{id_i}\}$.

Preceding studies such as [5, 12] and therein references have showed that β lies in the interval $[1 - r_{\max}, 1)$ where $r_{\max} = \frac{d_{\max}}{D}$ and $d_{\max} = \max_i d_i$.

The proposition 4 is a formal alternative proof for the lower bound on β stated in [5].

Proposition 4 *For any PRVP* **[P2]** *with* $(\mathbf{d}, D)$*,*

$$\beta \geqslant 1 - r_{\max}. \tag{24}$$

Proof. Using a (min or max) operation, we obtain

$$|\min_t \max_i x_{it} - \min_t \max_i tr_i| \leqslant \max_t |\max_i x_{it} - \max_i tr_i| \leqslant \max_t \max_i |x_{it} - tr_i|. \tag{25}$$

Since $\max_i r_i := r_{\max}$ is constant and $\min_t t = 1$,

$$|\min_t \max_i x_{it} - \min_t \max_i tr_i| = |\min_t \max_i x_{it} - r_{\max}|. \tag{26}$$

And it is clear that

$$\min_t \max_i x_{it} = \max_i \min_t x_{it} = 1. \tag{27}$$

Hence, from (25),(26) and (27), we conclude $\beta := \max_t \max_i |x_{it} - tr_i| \geqslant 1 - r_{\max}$. This completes the proof.

∎

In the following theorem, we propose a formal alternative proof for the interval in which t lies for each i and j.

Theorem 11 (The values of t). *Given the PRVP* **[P2]** *with* $(\mathbf{d}, D)$*, for each i and j*

$$t = \left\lceil \frac{j-\beta}{r_i} \right\rceil, \left\lceil \frac{j-\beta}{r_i} \right\rceil + 1, \ldots, \left\lfloor \frac{j-1+\beta}{r_i} + 1 \right\rfloor. \tag{28}$$

Proof. Consider a fixed product i and a fixed number j of a product i which have been allocated up to time (or position) t. That is $x_{it} = j$ such that $x_{it} > x_{i(t-1)} = j - 1$. We must show that

$$|x_{it} - tr_i| = |j - tr_i| \leqslant \beta \text{ and } |x_{i(t-1)} - (t-1)r_i| = |j - 1 - (t-1)r_i| \leqslant \beta. \tag{29}$$

From the inequality (29), we obtain

$$\frac{j-\beta}{r_i} \leqslant t \leqslant \frac{j+\beta}{r_i} \text{ and } \frac{j-1-\beta}{r_i} + 1 \leqslant t \leqslant \frac{j-1+\beta}{r_i} + 1. \tag{30}$$

Since $r_i < 1$ and $\beta > \frac{1-r_i}{2}$, the inequality (30) reduces to

$$\frac{j-\beta}{r_i} \leqslant t \leqslant \frac{j-1+\beta}{r_i} + 1. \tag{31}$$

In the inequality (31), if $t < \frac{j-\beta}{r_i}$, then $j - tr_i > \beta$, a contradiction to $|x_{it} - tr_i| \leqslant \beta$. Similarly, if $t > \frac{j-1+\beta}{r_i} + 1$, then $(t-1)r_i - (j-1) > \beta$, a contradiction to $|x_{i(t-1)} - (t-1)r_i| \leqslant \beta$. Therefore, we get

$$t = \left\lceil \frac{j-\beta}{r_i} \right\rceil, \left\lceil \frac{j-\beta}{r_i} \right\rceil + 1, \ldots, \left\lfloor \frac{j-1+\beta}{r_i} + 1 \right\rfloor. \tag{32}$$

This completes the proof.

■

Corollary 1 (Lower bound on β). *For any PRVP* **[P2]** *with* $(\mathbf{d}, D)$,

$$\beta \geqslant \max[1 - r_{\max}, \frac{1}{2}(1 - r_{\min})] \tag{33}$$

with equality if and only if $1 - r_{\max} \geqslant \frac{1}{2}(1 - r_{\min})$.

Proof. Theorem 11 shows that $\beta > \frac{1-r_i}{2}$ for all i. Namely, $\beta > \frac{1-r_{\min}}{2}$. Hence, by Proposition 4, we get the results as required.

■

Corollary 2 (Degree of a vertex $v_{ij} := (i, j)$). *In a bipartite graph* $G(\mathbf{d}, \beta^*)$ *corresponding to the PRVP* **[P2]** *with* $(\mathbf{d}, D)$, *for each i and j we have*

$$\deg(v_{ij}) = \left\lfloor \frac{j-1+\beta^*}{r_i} + 1 \right\rfloor - \left\lceil \frac{j-\beta^*}{r_i} \right\rceil + 1, \tag{34}$$

where $\beta^* := \min \beta$.

Proof. It is clear that $\deg(v_{ij}) = |\{t \mid x_{it} = j,\ x_{i(t-1)} = j-1,\ \beta = \beta^*\}|$. Hence, by Equation (32), we get

$$t = \left\lceil \frac{j-\beta^*}{r_i} \right\rceil, \left\lceil \frac{j-\beta^*}{r_i} \right\rceil + 1, \ldots, \left\lfloor \frac{j-1+\beta^*}{r_i} + 1 \right\rfloor . \tag{35}$$

This implies $\deg(v_{ij}) = \left\lfloor \frac{j-1+\beta^*}{r_i} + 1 \right\rfloor - \left\lceil \frac{j-\beta^*}{r_i} \right\rceil + 1$ as required.

■

The following two Corollaries 3, 4 are immediate from Corollary 1. We shall provide new results concerning a bipartite graph $G(\mathbf{d}, \beta^*)$, not stated in [5] and other previous studies.

Corollary 3 (Degree of a vertex $w_t := t$). *In a bipartite graph $G(\mathbf{d}, \beta^*)$ corresponding to the PRVP* [**P2**] *with* $(\mathbf{d}, D)$, $\deg(v_{ij})$ *is given by Corollary 2. Then for each t*

$$\deg(w_t) = \left| \left\{ i \,\middle|\, t \in \left[\left\lceil \frac{j-\beta^*}{r_i} \right\rceil, \left\lceil \frac{j-\beta^*}{r_i} \right\rceil + 1, \ldots, \left\lfloor \frac{j-1+\beta^*}{r_i} + 1 \right\rfloor \right] \right\} \right|, \tag{36}$$

where $\beta^* := \min\ \beta$.

Proof. Consider a fixed time (or position) t. Then $\deg(w_t)$ is a number of (i,j) pairs adjacent to a vertex w_t. Now that a graph is bipartite, in fact, $\deg(w_t)$ has been already determined by $\deg(v_{ij})$ in Corollary 2. Thus, $\deg(w_t)$ is a number of a product i which is allocated at t position. Hence, from Equation (32), we get the result as required.

■

Corollary 4 (Linearly ordered set $\mathcal{P}_i$). *In a bipartite graph $G(\mathbf{d}, \beta^*)$ corresponding to the PRVP* [**P2**] *with* $(\mathbf{d}, D)$, $\deg(v_{ij})$ *is given by Corollary 2. Then for each i and t*

$$|\{j \mid x_{it} = j\}| = |\{j \mid v_{ij} \sim w_t\}| = 1. \tag{37}$$

And for each i, $(\mathcal{P}_i, \precsim)$ is a linearly ordered set (i.e. chain) such that

$$\begin{aligned} v_{ij} \precsim v_{ik} &\quad \text{if and only if} \quad j \leqslant k \\ \text{or equivalently, } v_{ij} \precsim v_{ik} &\quad \text{if and only if} \quad t_1 \leqslant t_2, \\ \text{where } (v_{ij} \sim w_{t_1}) &\quad \text{and} \quad (v_{ik} \sim w_{t_2}). \end{aligned} \tag{38}$$

Proof. We consider a fixed product i and a fixed time (or position) t. Assume that $|\{j \mid x_{it} = j\}| \geqslant 2$. Then there exists at least one element, say, $j+1$ that satisfies Equation (37). That is, $x_{it} = j+1$. From Equations (31) and (32), we get $t = \left\lceil \frac{j+1-\beta^*}{r_i} \right\rceil, \left\lceil \frac{j+1-\beta^*}{r_i} \right\rceil + 1, \ldots, \left\lfloor \frac{j+\beta^*}{r_i} + 1 \right\rfloor$. Note that $\left\lceil \frac{j+1-\beta^*}{r_i} \right\rceil > \left\lfloor \frac{j-1+\beta^*}{r_i} + 1 \right\rfloor$. This is a contradiction to $|\{j \mid x_{it} = j\}| \geqslant 2$. This completes the proof.

It is obvious that a relation $\precsim$ satisfies a reflexive, an antisymmetric and a transitive properties. It suffices to confirm that a relation $\precsim$ is well-defined. Note that $v_{ij} \sim w_{t_1}$ if and only if $x_{it_1} = j$ and $v_{ik} \sim w_{t_2}$ if and only if $x_{it_2} = k$. From the result of $\left|\{j \mid x_{it} = j\}\right| = 1$, $j \neq k$ if and only if $t_1 \neq t_2$. In other words, $t_1 > t_2$ if and only if $j > k$ or $t_1 < t_2$ if and only if $j < k$. This completes the proof.

■

Remark 1. The lower bound of β given by Corollary 1 is obviously refined compared to Steiner et al. ([5]) and Balinski et al. ([12]).

Remark 2. [A convex bipartite graph $G(\mathbf{d}, \beta^*)$] The graph $G(\mathbf{d}, \beta^*)$ corresponding to the PRVP **[P2]** is established as follows.

(1) Based on the n and $\mathbf{d}$, calculate $r_{\max}$.

(2) Using Equation (33), find the interval of α. †

(3) Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$. †

For $b = 1, 2, \ldots k$,

(4) Set $\beta_b := \frac{\alpha_b}{D}$, where $D := \sum_{i=1}^{n} d_i$.

(5) Calculate the values of t for each i and j by Equation (28).

(6) Join $v_{ij} := (i, j)$ and $w_t := t$.

(7) Obtain the bipartite adjacency matrix $\tilde{\mathbf{X}}$ of a graph $G(\mathbf{d}, \beta_b)$.

(8) Check whether or not there exists a perfect matching in the graph $G(\mathbf{d}, \beta_b)$ by the $per(\tilde{\mathbf{X}})$. If exists, then $\beta_b = \beta^*$ and terminate. Otherwise, then set $b \leftarrow b + 1$ and goto (4).

4.2 Convexity of a graph $G(\mathbf{d}, \beta^*)$

Proposition 5. *The graph $G(\mathbf{d}, \beta^*)$ is always convex with respect to the vertex set $\mathcal{W}$ but is generally nonconvex concerning the vertex set $\mathcal{V}$.*

Proof. (1) Always convex w.r.t. $\mathcal{W}$: Obvious from Definition 4, Equations (34) and (35) in Corollary 2.

(2) Generally nonconvex w.r.t. $\mathcal{V}$: From Proposition 3 and Corollary 3, it is evident that

$$1 \leqslant \deg w_t \leqslant n.$$

We consider two cases.

Case 1. $\deg w_t = 1, \ \forall\, t$.

This case indicates $\mathrm{per}(\tilde{X}) = |\mathcal{M}| = 1$. Thus, the graph $G(\mathbf{d}, \beta^*)$ is convex with respect to the vertex set $\mathcal{V}$ by Definition 4.

Case 2. $\exists\, t$ such that $\deg w_t = \omega \neq 1$.

†See Equation (39)

†See Equation (40)

Assume that the graph $G(\mathbf{d}, \beta^*)$ is convex with respect to the vertex set $\mathcal{V}$. Then from Definition 4 of a convex bipartite graph, the vertex set $\mathcal{V}$ must be linearly or totally ordered set with a certain relation $R := \precsim$. Note that from Corollary 4, for each i, $\mathcal{P}_i$ is a chain in $\mathcal{V} = \bigcup_{i=1}^{n} \mathcal{P}_i$ such that $\mathcal{P}_i \bigcap \mathcal{P}_j = \varnothing$ for any $i \neq j$. This implies a vertex set $\mathcal{V}$ should be a partially ordered set *without* a certain relation $R := \precsim$. By Corollary 4 and the assumption, we obtain

$$\begin{aligned}
&(i) \quad t \sim \Big\{(i_1, j_1),\, (i_2, j_2),\, \cdots,\, (i_\omega, j_\omega)\Big\},\ 2 \leqslant \omega \leqslant n,\\
&(ii) \quad (i_1, j_1) \precsim (i_2, j_2) \precsim \cdots \precsim (i_\omega, j_\omega),\\
&(iii) \quad i_1 \neq i_2 \neq \cdots \neq i_\omega.
\end{aligned}$$

Then by Definition 4, it is required that

$$\begin{aligned}
t \ \sim\ & \Big\{(i_1, j_1),\, (i_2, j_2),\, \cdots,\, (i_\omega, j_\omega)\Big\} \bigcup \\
& \Big\{(i_1, j_1+1),\, (i_1, j_1+2),\, \cdots,\, (i_1, d_{i_1}),\, \cdots,\, (i_2, j_2-1),\\
& (i_2, j_2+1),\, (i_2, j_2+2),\, \cdots,\, (i_2, d_{i_2}),\, \cdots,\, (i_\omega, j_\omega-1)\Big\},
\end{aligned}$$

which explains $\deg(w_t) > \omega$. This is a contradiction to $\deg(w_t) = \omega$.
Conclusively, the graph $G(\mathbf{d}, \beta^*)$ is generally nonconvex with respect to the vertex set $\mathcal{V}$.

∎

Remark 3. The graph $G(\mathbf{d}, \beta^*)$ is convex with respect to the vertex set $\mathcal{V}$ if and only if $\mathrm{per}(\tilde{X}) = |\mathcal{M}| = 1$.

4.3 Bijection between optimal sequences and perfect matchings

In this subsection, we shall show that there exists a bijection between the set $\mathcal{S}$ of optimal sequences and the set $\mathcal{M}$ of perfect matchings in the convex bipartite graph $G(\mathbf{d}, \beta^*)$. Also, we shall arrange graph theoretic necessary and sufficient conditions for the optimality, i.e., the existence of perfect matching.

Theorem 12. *In the PRVP* **[P2]** *with* $(\mathbf{d}, D)$ *and its corresponding graph* $G(\mathbf{d}, \beta^*)$, *we have* $|\mathcal{S}| = |\mathcal{M}| = \mathrm{per}(\tilde{\mathbf{X}})$.

Proof. From Equation (33), we get $\beta^* := \frac{\alpha^*}{D}$ such that

$$\begin{aligned}
&\alpha^* \in [D - d_{\max}, D - d_{\max} + 1, \ldots, D-1]\,, if\ 1 - r_{\max} \geqslant \frac{1}{2}(1 - r_{\min})\\
&\alpha^* \in [\lceil 0.5(D - d_{\min}) \rceil\,, \lceil 0.5(D - d_{\min}) \rceil + 1, \ldots, D-1]\,, otherwise.
\end{aligned} \tag{39}$$

This means the maximum numbers of iteration are

$$d_{\max}\ ,\ if\ 1 - r_{\max} \geqslant \frac{1}{2}(1 - r_{\min})$$

$$\lfloor 0.5(D + d_{\min}) \rfloor \text{ , } otherwise. \tag{40}$$

Thus, we can find α^* and β^* with a finite iteration. In other words, we can obtain an optimal sequence in the PRVP **[P2]** at all times. Then, by Equation (21), we get an edge $\epsilon_{s_t jt} := v_{s_t j} \sim w_t$ such that $j = x_{s_t t}$, for all s_t, $t = 1, \ldots, D$. Namely, an optimal sequence implies a perfect matching. Now, suppose that there is a perfect matching in the graph $G(\mathbf{d}, \beta^*)$. Then, in this perfect matching, we can find $\sum_{i=1}^{n} x_{it} = t$ for all t. This fact is the definition of an optimal sequence. Already, we have confirmed that there exists an optimal sequence. Therefore, we conclude that a perfect matching in the graph $G(\mathbf{d}, \beta^*)$ can be ensured at all times. Conclusively, there exists a bijection between the set $\mathcal{S}$ of optimal sequences and the set $\mathcal{M}$ of perfect matchings in the convex bipartite graph $G(\mathbf{d}, \beta^*)$. That is, $\mathcal{S} \backsimeq \mathcal{M}$. And it is evident that $|\mathcal{S}| = |\mathcal{M}| = \text{per}(\tilde{\mathbf{X}})$ by Theorem 9.

■

Proposition 6. *For any PRVP* **[P2]** *with* $(\mathbf{d}, D)$ *and its corresponding graph* $G(\mathbf{d}, \beta^*)$*, the following statements are equivalent.*

(1) A sequence is optimal.

(2) There exists a perpect matching.

(3) A vertex covering number of $G(\mathbf{d}, \beta^*)$ *is equal to a size of maximum matching of* $G(\mathbf{d}, \beta^*)$*.*

(4) A size of maximum matching of $G(\mathbf{d}, \beta^*)$ *is equal to* $D := \sum_{i=1}^{n} d_i$*.*

(5) The permanent of a bipartite adjacency matrix $\tilde{\mathbf{X}}$ *of* $G(\mathbf{d}, \beta^*)$ *is greater than or equal to 1.*

Proof. Since $G(\mathbf{d}, \beta^*)$ is a bipartite graph, we obtain the following result by Theorem 10, Proposition 2 and Proposition 3.

$$2 \leqslant \Delta(G) \leqslant \chi_I(G), \delta(G) \geqslant 1. \tag{41}$$

Thus, from Theorem 4 and Theorem 5, it is clear that

$$\alpha(G) + \tau(G) = 2D, \; \nu(G) + \rho(G) = 2D, \; \tau(G) = \nu(G), \; \alpha(G) = \rho(G). \tag{42}$$

On the other hand, we see that

$$|\mathcal{M}| \neq 0 \Leftrightarrow |N(\mathcal{V}_s)| \geqslant |\mathcal{V}_s|, \forall \mathcal{V}_s \subseteq \mathcal{V} \Leftrightarrow |\mathcal{V}| = |\mathcal{W}| = \tau(G) = \nu(G) = D \tag{43}$$

by Theorem 6, Theorem 7 and Theorem 8. This implies that

$$(2) \Leftrightarrow (3) \Leftrightarrow (4). \tag{44}$$

And Theorem 9 gives us

$$|\mathcal{M}| \neq 0 \Leftrightarrow \text{per}(\tilde{\mathbf{X}}) \geqslant 1. \tag{45}$$

Therefore,

$$(2) \Leftrightarrow (5). \tag{46}$$

Finally, from Theorem 12, it is obvious that

$$(1) \Leftrightarrow (2). \tag{47}$$

Conclusively, combining all equations (44), (46), and (47) in one, we have the results as required.

∎

4.4 Examples

In this section, we consider seven examples. As aforementioned in the previous sections 1, 2 and 3, we can apply three approaches to the PRVP as follows.

- Integer programming approach (abbreviated to IP approach),
- Balanced sequence approach (abbreviated to BS approach) and in discussion,
- Convex bipartite graph approach (abbreviated to CBG approach).

Through these examples, we would like to claim many advantages for the CBG approach in comparison with other approaches.

4.4.1 The case $n = 2$

Example 2. Consider the instance $\mathbf{d} = (3, 2)$. We shall apply IP and CBG approaches to this example.

Table 2 gives an optimal solution for x_{it}'s and β which are numerically analyzed by the method of the integer programming. Table 2 shows that an optimal sequence is $\mathbf{s} =$

Table 2: IP approach : x_{it}'s optimal values under $\mathbf{d} = (3, 2)$ and $\beta^* = \frac{2}{5}$

$i \backslash t$	1	2	3	4	5
1	1	1	2	2	3
2	0	1	1	2	2

$(1, 2, 1, 2, 1)$, since $x_{11} > x_{10}$, $x_{22} > x_{21}$, $x_{13} > x_{12}$, $x_{24} > x_{23}$, $x_{15} > x_{14}$.

By Remark 2, we obtain the edge set $E = \{e_{11,1}, e_{12,3}, e_{13,5}, e_{21,2}, e_{22,4}\}$ and $\beta^* = \frac{2}{5}$. And the bipartite adjacency matrix is given by

$$\tilde{\mathbf{X}}_{5\times 5} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (1,2) \\ (2,2) \\ (1,3) \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Thus, we get

$$|\mathcal{M}| = \text{per}(\tilde{\mathbf{X}}) = 1, \ \det(\tilde{\mathbf{X}}) = 1, \ \text{rank}(\tilde{\mathbf{X}}) = 5$$

and the optimal sequence $\mathbf{s} = (1,2,1,2,1)$. Note that this optimal sequence is *balanced and periodic* from Theorem 1 and Lemma 1.

Example 3. Consider the instance $\mathbf{d} = (6,4) = 2\cdot(3,2)$. We shall apply CBG approach to this example.

In the same manner, by Remark 2, we get the edge set

$$E = \{e_{11,1}, e_{12,3}, e_{13,5}, e_{14,6}, e_{15,8}, e_{16,10}, e_{21,2}, e_{22,4}, e_{23,7}, e_{24,9}\}$$

and $\beta^* = \frac{4}{10}$. Thus,

$$|\mathcal{M}| = \text{per}(\tilde{\mathbf{X}}) = 1.$$

The optimal sequence is $\mathbf{s} = (1,2,1,2,1,1,2,1,2,1)$, which is *periodic* by Lemma 1. In general, if $\mathbf{d} = n\cdot(3,2)$, then the unique optimal sequence can be described as

$$\overbrace{(1,2,1,2,1)(1,2,1,2,1)\cdots(1,2,1,2,1)}^{n \ \ times}.$$

4.4.2 The case $n = 3$

The following two examples are based on the Theorem 2 and Lemma 1.

Example 4. Consider the instance $\mathbf{d} = (4,2,1)$. We shall apply IP, BS and CBG approaches to this example, respectively.

In Table 3, it is clear that the optimal sequence is $\mathbf{s} = (1,2,1,3,1,2,1)$.

Table 3: IP approach : x_{it}'s optimal values under $\mathbf{d} = (4,2,1)$ and $\beta^* = \frac{3}{7}$

$i \backslash t$	1	2	3	4	5	6	7
1	1	1	2	2	3	3	4
2	0	1	1	1	1	2	2
3	0	0	0	1	1	1	1

In BS approach, the optimal sequence has been reported by $\mathbf{s} = (1,2,1,3,1,2,1)$ ([3, see Theorem 2.19.]).

And from Remark 2, the edge set is

$$E = \{e_{11,1}, e_{12,3}, e_{13,5}, e_{14,7}, e_{21,2}, e_{22,6}, e_{31,4}\}$$

with $\beta^* = \frac{3}{7}$. In addition, we confirm that the bipartite adjacency matrix is

$$\tilde{\mathbf{X}}_{7\times7} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (1,2) \\ (3,1) \\ (2,2) \\ (1,3) \\ (1,4) \end{array} \!\!\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Hence,

$$|\mathcal{M}| = \text{per}(\tilde{\mathbf{X}}) = 1, \;\; \det(\tilde{\mathbf{X}}) = -1, \;\; \text{rank}(\tilde{\mathbf{X}}) = 7.$$

Likewise, the optimal sequence is $\mathbf{s} = (1, 2, 1, 3, 1, 2, 1)$.

Example 5. Consider the instance $\mathbf{d} = (3, 3, 1)$.

In section 3, we have already stated the optimal solution by IP approach: the optimal sequence is $\mathbf{s} = (2, 1, 1, 2, 3, 2, 1)$.

Now, let us consider the CBG approach. By Remark 2, we have the edge set

$$\begin{aligned} E \;\; = \;\; & \{e_{11,1}, e_{11,2}, e_{12,3}, e_{12,4}, e_{12,5}, e_{13,6}, e_{13,7}, e_{21,1}, e_{21,2}, \\ & e_{22,3}, e_{22,4}, e_{22,5}, e_{23,6}, e_{23,7}, e_{31,2}, e_{31,3}, e_{31,4}, e_{31,5}, e_{31,6}\} \end{aligned}$$

and $\beta^* = \frac{5}{7}$. From this edge set, the bipartite adjacency matrix is given by

$$\tilde{\mathbf{X}}_{7\times7} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (1,2) \\ (3,1) \\ (2,2) \\ (1,3) \\ (2,3) \end{array} \!\!\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

And by this bipartite adjacency matrix, we have

$$|\mathcal{M}| = \text{per}(\tilde{\mathbf{X}}) = 24,$$

which explains that the number of all the perfect matchings (i.e. optimal sequences) is 24 in this example. For further particulars, see Table 4. In Table 4, $\sum\sum_{i,t}|x_{it} - tr_i|$ denotes the *total deviation.* Note that there exist eight sequences which minimize the total deviation.

Table 4: CBG approach : All the optimal sequences (i.e. perfect matchings) under $\mathbf{d} = (3,3,1)$ and $\beta^* = \frac{5}{7}$

$\sum\sum_{i,t} \lvert x_{it} - tr_i \rvert$	1	2	3	4	5	6	7
17/7	1	2	1	2	3	1	2
17/7	1	2	1	2	3	2	1
15/7	1	2	1	3	2	1	2
15/7	1	2	1	3	2	2	1
17/7	1	2	2	1	3	1	2
17/7	1	2	2	1	3	2	1
15/7	1	2	2	3	1	1	2
15/7	1	2	2	3	1	2	1
13/7	1	2	3	1	2	1	2
13/7	1	2	3	1	2	2	1
13/7	1	2	3	2	1	1	2
13/7	1	2	3	2	1	2	1
17/7	2	1	1	2	3	1	2
17/7	2	1	1	2	3	2	1
15/7	2	1	1	3	2	1	2
15/7	2	1	1	3	2	2	1
17/7	2	1	2	1	3	1	2
17/7	2	1	2	1	3	2	1
15/7	2	1	2	3	1	1	2
15/7	2	1	2	3	1	2	1
13/7	2	1	3	1	2	1	2
13/7	2	1	3	1	2	2	1
13/7	2	1	3	2	1	1	2
13/7	2	1	3	2	1	2	1

4.4.3 The case $n = 4$

The following three examples are on the ground of Theorem 3, Proposition 1 and Lemma 1.

Example 6. Consider the instance $\mathbf{d} = (8,4,2,1)$. We shall apply CBG, BS and IP approaches to this example, respectively.

By remark 2, we have the edge set

$$\begin{aligned} E &= \{e_{11,1}, e_{12,3}, e_{13,5}, e_{14,7}, e_{15,9}, e_{16,11}, e_{17,13}, \\ &\quad e_{18,15}, e_{21,2}, e_{22,6}, e_{23,10}, e_{24,14}, e_{31,4}, e_{32,12}, e_{41,8}\} \end{aligned}$$

and $\beta^* = \frac{7}{15}$. And we obtain

$$|\mathcal{M}| = \mathrm{per}(\tilde{\mathbf{X}}) = 1,\ \det(\tilde{\mathbf{X}}) = -1,\ \mathrm{rank}(\tilde{\mathbf{X}}) = 15.$$

Therefore, the optimal sequence is given by

$$\mathbf{s} = (1,2,1,3,1,2,1,4,1,2,1,3,1,2,1).$$

On the other hand, the same optimal sequence

$$\mathbf{s} = (1,2,1,3,1,2,1,4,1,2,1,3,1,2,1)$$

has been given by [3, see Theorem 2.20.].

Table 5 gives an optimal solution for x_{it}'s and β which are numerically analyzed by the method of the integer programming. Of course, the optimal sequence is the same as the one in CBG and BS approaches.

Table 5: IP approach : x_{it}'s optimal values under $\mathbf{d} = (8,4,2,1)$ and $\beta^* = \frac{7}{15}$

$i \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8
2	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4
3	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2
4	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Example 7. We consider the instance $\mathbf{d} = (2,1,1,1)$ by the approaches of IP and CBG.

Table 6 gives an optimal solution for x_{it}'s and β. In Table 6, we find the optimal sequence $\mathbf{s} = (1,3,2,4,1)$.

Table 6: IP approach : x_{it}'s optimal values under $\mathbf{d} = (2,1,1,1)$ and $\beta^* = \frac{3}{5}$

$i \backslash t$	1	2	3	4	5
1	1	1	1	1	2
2	0	0	1	1	1
3	0	1	1	1	1
4	0	0	0	1	1

In CBG approach, we have the edge set

$$E = \{e_{11,1}, e_{11,2}, e_{12,4}, e_{12,5}, e_{21,2}, e_{21,3}, e_{21,4}, e_{31,2}, e_{31,3}, e_{31,4}, e_{41,2}, e_{41,3}, e_{41,4}\}$$

and $\beta^* = \frac{2}{5}$. The bipartite adjacency matrix is as follows.

$$\tilde{\mathbf{X}}_{5\times 5} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (1,2) \\ (3,1) \\ (4,1) \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Hence,

$$|\mathcal{M}| = \mathrm{per}(\tilde{\mathbf{X}}) = 6.$$

And the optimal sequences are given by

$$(1,2,3,4,1),\ (1,4,2,3,1),\ (1,3,4,2,1),(1,4,3,2,1),\ (1,2,4,3,1),\ (1,3,2,4,1).$$

All of the optimal sequences have a same total deviation value of $\frac{9}{5}$.

Example 8. Finally, let us consider the instance $\mathbf{d} = (4,4,2,1)$ by the approaches of CBG and BS.

To begin with, by Remark 2, we have the edge set

$$\begin{aligned} E \;=\; & \{e_{11,1}, e_{11,2}, e_{12,4}, e_{12,5}, e_{13,7}, e_{13,8}, e_{14,10}, e_{14,11}, e_{21,1}, e_{21,2}, \\ & e_{22,4}, e_{22,5}, e_{23,7}, e_{23,8}, e_{24,10}, e_{24,11}, e_{31,2}, e_{31,3}, e_{31,4}, e_{32,8}, \\ & e_{32,9}, e_{32,10}, e_{41,4}, e_{41,5}, e_{41,6}, e_{41,7}, e_{41,8}\} \end{aligned}$$

and $\beta^* = \frac{7}{11}$. The bipartite adjacency matrix is described as

$$\tilde{\mathbf{X}}_{11\times11} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (1,2) \\ (3,1) \\ (2,2) \\ (1,3) \\ (4,1) \\ (3,2) \\ (2,3) \\ (1,4) \\ (2,4) \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Thus, we have

$$|\mathcal{M}| = \mathrm{per}(\tilde{\mathbf{X}}) = 16.$$

And the optimal sequences are given by Table 7.

In BS approach, Altman et al. ([3, Appendix]) has reported the optimal sequence of the above example as follows:

$$(1,2,3,1,2,1,2,3,1,2,4).$$

Regrettably, the sequence they have stated is by no means optimal. Consider deviations for each time (or position). They are given by

$$\frac{7}{11}, \frac{3}{11}, \frac{5}{11}, \frac{6}{11}, \frac{2}{11}, \underline{\frac{9}{11}}, \frac{5}{11}, \frac{6}{11}, \underline{\frac{8}{11}}, \frac{4}{11}, \frac{0}{11}.$$

Table 7: CBG approach : All the optimal sequences (i.e. perfect matchings) under $\mathbf{d} = (4, 4, 2, 1)$ and $\beta^* = \frac{7}{11}$

$\sum\sum_{i,t} \lvert x_{it} - tr_i \rvert$	1	2	3	4	5	6	7	8	9	10	11
42/11	1	2	3	1	2	4	1	2	3	1	2
42/11	1	2	3	1	2	4	1	2	3	2	1
42/11	1	2	3	1	2	4	2	1	3	1	2
42/11	1	2	3	1	2	4	2	1	3	2	1
42/11	1	2	3	2	1	4	1	2	3	1	2
42/11	1	2	3	2	1	4	1	2	3	2	1
42/11	1	2	3	2	1	4	2	1	3	1	2
42/11	1	2	3	2	1	4	2	1	3	2	1
42/11	2	1	3	1	2	4	1	2	3	1	2
42/11	2	1	3	1	2	4	1	2	3	2	1
42/11	2	1	3	1	2	4	2	1	3	1	2
42/11	2	1	3	1	2	4	2	1	3	2	1
42/11	2	1	3	2	1	4	1	2	3	1	2
42/11	2	1	3	2	1	4	1	2	3	2	1
42/11	2	1	3	2	1	4	2	1	3	1	2
42/11	2	1	3	2	1	4	2	1	3	2	1

And a total deviation is $\frac{55}{11}$. Note that the deviations of $t = 6, 9$ are a contradiction to $\beta^* = \frac{7}{11}$.

5 Concluding remarks

5.1 Summary and contribution

The PRVP is to find a sequence that minimizes the maximum value of a deviation function between ideal and actual demands.

The PRVP is an important scheduling problem that arises on mixed-model assembly lines. And it is possible for us to exhibit the link between balanced sequences (word combinatorics) and this scheduling problem.

The objective of this study is to show that we can provide a more efficient and in-depth procedure with a graph theoretic approach, not an integer programming one, in order to solve the PRVP. To achieve this goal, we propose formal alternative proofs for some of the results stated in [5], and establish several new results and problems in terms of graph theory.

The very core of our methodology is to establish the convex bipartite graph $G(\mathbf{d}, \beta^*)$ corresponding to the PRVP. Then, the existence and the number of perfect matchings in the graph $G(\mathbf{d}, \beta^*)$ (i.e. optimal sequences in the PRVP) can be confirmed by the permanent of the bipartite adjacency matrix.

We give the outline of our study as follows.

- We have proposed two formal alternative proofs for the results stated in [5]: the lower bound of β^* and the degree of vertex v_{ij}. (see Proposition 4, Theorem 11 and Corollary 2)
- Also, we have provided two new results not stated in [5, 12]: the degree of vertex w_t and the refined lower bound of β^*. (see Corollary 3, 4 and 1)
- We have introduced a procedure for the construction of the graph $G(\mathbf{d}, \beta^*)$. (see Remark 2)
- We have proved that the graph $G(\mathbf{d}, \beta^*)$ is always convex with respect to the vertex set $\mathcal{W}$. (see Proposition 5 and Remark 3)
- We have confirmed that an optimal sequence corresponds to a perfect matching in the graph $G(\mathbf{d}, \beta^*)$. That is, $|\mathcal{S}| = |\mathcal{M}|$ and there exists a bijection from $\mathcal{S}$ to $\mathcal{M}$. This implies that an optimal sequence in a integer programming approach is equivalent to a perfect matching in a graph theoretic approach. (see Theorem 12)
- Finally, we have established several new problems of the PRVP in terms of a graph theory. (see Proposition 6)

And the contributions of our study to the PRVP are as follows.

Remark 4. Our lower bound of β given by Corollary 1 is obviously refined compared to Steiner et al. ([5]) and Balinski et al. ([12]). So it is required only a few iterations to solve the PRVP, which explains

$$\begin{aligned}\text{Number of iterations} &= d_{\max}, \text{ if } 1 - r_{\max} \geqslant \frac{1}{2}(1 - r_{\min}) \\ \text{Number of iterations} &= \lfloor 0.5(D + d_{\min}) \rfloor, \text{ otherwise}\end{aligned}$$

Remark 5. Our study provides an efficient and exact solution to the minimization problem of a total distance (or deviation), abbreviated the MPTD, since the optimal solutions to the MPTD exist in the optimal solutions to the PRVP. That is, for the set of optimal solutions to the MPTD, denoted by $\tilde{\mathcal{S}}$, there exists an injection from $\tilde{\mathcal{S}}$ to $\mathcal{S}$.

5.2 Further studies

First of all, the CBG approach shall be applied to the MPTD. This is only the extension of the PRVP to the MPTD. Hence, it may be not a difficult work.

With respect to the Remark 4, we conjecture that for any n, there exists a closed form solution to the MPMD such that $d_{\max} \geqslant 3$. Note that there exists a closed form solution to the MPMD in case of $d_{\max} = 2$. To achieve this goal, we would like to suggest the use of the characteristic polynomial of an adjacency matrix or a bipartite adjacency matrix and the study of the Fraenkel conjecture.

Finally, we shall investigate the area to which the CBG approach can be applied, such as scheduling problems, queueing network, word combinatorics and so forth.

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