한국소음진동공학회 2004년도 추계학술대회논문집, pp. 140~144.

Global Bifurcations and Chaos in an Harmonically Excited and Undamped Circular Plate

Sergey B. Samoylenko and Won Kyoung Lee

Key words: nonlinear vibration of circular plate, global bifurcation, Melnikov's method.

ABSTRACT

Global bifurcations and chaos in modal interactions of an imperfect circular plate with one-to-one internal resonance are investigated. The case of primary resonance, in which an excitation frequency is near natural frequencies, is considered. The damping force is not included in the analysis. The Melnikov's method for heteroclinic orbits of the autonomous system was used to obtain the criteria for chaotic motion.

1. Introduction

Dynamical systems having two of their linear natural frequencies nearly equal exhibit complicated and interesting phenomena when nonlinear terms are taken into account. Plates may be one of the systems. The dynamics of plates related with non-linear modal interactions was widely described in terms of local bifurcations by Sridhar and Mook [1], Sethna et al. [2]. Lee et al. [3, 4] studied modal interactions of a circular plate and of the plate on an elastic foundation [5, 6] for which internal resonance occurs. Global bifurcations have been examined for a wide class of problems. Feng and Sethna [7] studied global bifurcations of a Hamiltonian system with a certain symmetry in terms of breaking of heteroclinic orbits. Yeo and Lee [8] studied global bifurcations in modal interactions of a circular plate using method developed by Wiggins and Kovačič [9]. They investigated homoclinic orbit created in a resonance resulting from perturbation.

In this study we extended Yeo and Lee's work [8] to investigate heteroclinic orbits created in a non-resonance case. In order to simplify the problem we consider the undamped system. Melnikov method [10, 11] was used to study global bifurcations due to breaking of homoclinic orbits.

* School of Mechanical Engineering, Yeungnam Uni.

E-mail: wklee@yu.ac.kr

2. Governing equations

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [10]. Yeo and Lee [8] simplified these equations to fit the special case of non-uniform circular plate shown in Fig. 1, for which forcing terms were added.

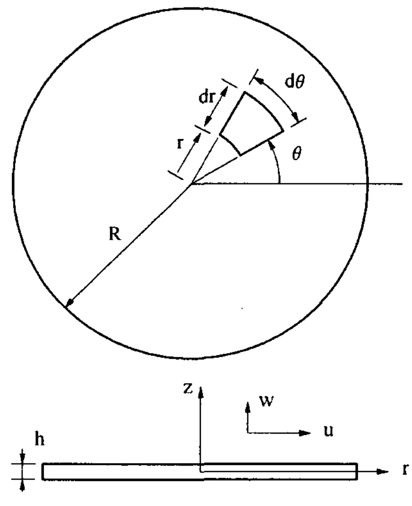


Fig. 1. A schematic diagram of a circular plate.

They assumed that transverse

displacement of the plate could be expressed as a combination of two linearized modes. Neglecting damping terms we reduce equations obtained by Yeo and Lee as follows:

$$\ddot{x}_i + \omega_i^2 x_i + \varepsilon\gamma\omega_i^2 x_i (x_1^2 + x_2^2) = \varepsilon\mu_i \cos\lambda t, \quad i = 1,2 \,, \tag{1}$$

where x_i are amplitudes of normal modes, ω_i are normal frequencies, γ plays role of the parameter of nonlinearity, μ_i are amplitudes of excitation and ε is a small parameter.

In order to consider internal resonance due to imperfection, $\omega_1 \approx \omega_2$, and external resonance due to forcing, $\lambda \approx \omega_1$, we introduce two parameters β and σ as follows:

$$\omega_2 = \omega_1 + \varepsilon\beta, \ \omega_1 = \lambda + \varepsilon\sigma \,,$$

where β and σ are called internal and external detuning parameters, respectively.

By use of the method of multiple scales [11] we get

$$Z'_1 = i[\varepsilon F_1 + \sigma Z_1 + \frac{3}{2}\gamma\lambda Z_1^2 Z_1^* + \gamma\lambda Z_1 Z_2 Z_2^* + \frac{1}{2}\gamma\lambda Z_2^2 Z_1^*] \tag{2a}$$

$$Z'_2 = i[\varepsilon F_2 + (\sigma+\beta) Z_2 + \frac{3}{2}\gamma\lambda Z_2^2 Z_2^* + \gamma\lambda Z_2 Z_1 Z_1^* + \frac{1}{2}\gamma\lambda Z_1^2 Z_2^*] \tag{2b}$$

where Z_i is slow-time amplitude:

$$x_{i0} = Z_i(T_1) e^{i\lambda T_0} + Z_i^*(T_1) e^{-i\lambda T_0},$$
$$T_i = \varepsilon^i t, \quad i = 0,1,\dots \,.$$

Here asterisk denotes complex conjugate, a prime denotes differentiation with respect to slow time, and $F_{1,2}$ are forcing terms: $F_i = -\mu_i / 4\varepsilon\lambda$.

Assuming harmonic amplitudes Z_i as $Z_i = \sqrt{2I_i}(\sin\theta_i + i\cos\theta_i)$, and making variable change with rescaling:

$$p_1 = I_1\gamma\lambda, \qquad q_1 = \theta_1 - \theta_2$$
$$p_2 - p_1 = I_2\gamma\lambda, \qquad q_2 = \theta_2 - \pi/2$$
$$F_1 = -f_1/\sqrt{\gamma\lambda}, \quad F_2 = f_2/\sqrt{\gamma\lambda}$$

we transform the system (2) to

$$P'_1 = 2P_1(P_2 - P_1)\sin 2Q_1 + \varepsilon f_1\sqrt{2P_1}\cos(Q_1+Q_2) = -\frac{\partial H_\varepsilon}{\partial Q_1} \tag{3a}$$

$$Q'_1 = \beta - (1+\cos 2Q_1)(2P_1 - P_2) + \varepsilon\left[\frac{f_2}{\sqrt{2(P_2-P_1)}}\cos Q_2 - \frac{f_1}{\sqrt{2P_1}}\sin(Q_1+Q_2)\right] = \frac{\partial H_\varepsilon}{\partial P_1} \tag{3b}$$

$$P'_2 = \varepsilon\left[f_1\sqrt{2P_1}\cos(Q_1+Q_2) - f_2\sqrt{2(P_2-P_1)}\sin Q_2\right] = -\frac{\partial H_\varepsilon}{\partial Q_2} \tag{3c}$$

$$Q'_2 = -\beta - \sigma + (1+\cos 2Q_1)P_1 - 3P_2 - \varepsilon\frac{f_2}{\sqrt{2(P_2-P_1)}}\cos Q_2 = \frac{\partial H_\varepsilon}{\partial P_2} \tag{3d}$$

with

$$H_\varepsilon = \beta P_1 - (\beta+\sigma)P_2 - (1+\cos 2Q_1)P_1(P_1 - P_2) - \frac{3}{2}P_2^2 - \frac{\varepsilon f_1}{\sqrt{2P_1}}\sin(Q_1+Q_2) - \frac{\varepsilon f_2}{\sqrt{2(P_2-P_1)}}\cos Q_2$$

3. Unperturbed system

Let us consider an unperturbed system, which corresponds to case $\varepsilon = 0$ in system (3) as follows:

$$P'_1 = 2P_1(P_2 - P_1)\sin 2Q_1 \tag{4a}$$
$$Q'_1 = \beta - (1+\cos 2Q_1)(2P_1 - P_2) \tag{4b}$$
$$P'_2 = 0 \tag{4c}$$
$$Q'_2 = -\beta - \sigma + (1+\cos 2Q_1)P_1 - 3P_2 \,. \tag{4d}$$

System (4) is a completely integrable Hamiltonian system with $P_2 = P_{20}$ as a conserved quantity. In view of the structure of system (4) study of the flow in two-dimensional space (P_1, Q_1) may be useful to understand the system. It has five fixed points, whose locations depend on two parameters β and P_{20}:

$$P_1^* = \frac{1}{4}(\beta + 2P_{20}),\ Q_1^* = \pi n \qquad (5a)$$

$$P_1^* = 0,\ \ Q_1^* = 2\pi n \pm \tilde{Q}_1 \qquad (5b)$$

$$P_1^* = P_{20},\ Q_1^* = 2\pi n \pm \tilde{Q}_1 \qquad (5c)$$

where

$$\tilde{Q}_1 = \arccos\left[\sqrt{|\beta| / 2P_{20}}\right].$$

Analysis of stability of system (4a,b) shows that point (5a) is center and points (5b,c) are saddle points. Fixed points in two-dimensional system (4a,b) correspond to invariant tori in four-dimensional system (4).

Returning to coordinates $(I_1, \theta_1, I_2, \theta_2)$ gives us physical meaning of this solution in terms of plate oscillations. Center and saddle-points are turned out to be mixed-mode and single-mode solutions, respectively.

For the system (4a,b) it is possible to study global bifurcations in the (β, P_{20}) parameter plane shown in Fig. 2, where exist four regions

$$\text{(I)}\quad \beta < -2P_{20},\quad \text{(II)}\quad -2P_{20} < \beta < 0$$
$$\text{(III)}\quad 0 < \beta < 2P_{20},\ \text{(IV)}\quad 2P_{20} > \beta.$$

Each of these has qualitatively different behavior of the flow. For regions I and IV there are neither homoclinic nor heteroclinic orbits in (Q_1, P_1)-plane. For regions II and III each phase portrait contains three heteroclinic orbits (A, A', A'' in II and B, B', B'' in III). We will consider breaking of heteroclinic orbits under perturbation and consequences of this breaking. Therefore regions II and III will be of our interest.

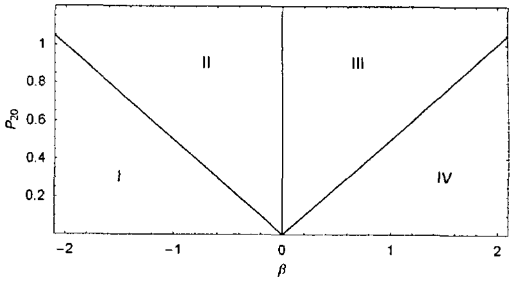


Fig. 2. Global bifurcation diagram of unperturbed system (18a,b).

4. Perturbed system and Melnikov theory

In order to investigate the behavior of the system under small perturbations we use generalized multidimensional Melnikov method [13]. In our case Melnikov integral has form:

$$M = \int_{-T_j^u}^{T_j^s} \frac{\partial \tilde{H}}{\partial Q_2} dt$$

Calculation of this integral along unperturbed heteroclinic orbits, gives following results.

For orbit A:

$$M^A = \frac{\sqrt{2P_{20}} \sin \tilde{Q}_1}{2e_A} m^A \cos[Q_{20} - \phi^A] \qquad (6)$$

where

$$m^A = m^A(f_1, f_2, \sigma, \beta, P_{20})$$
$$\phi^A = \phi^A(f_1, f_2, \sigma, \beta, P_{20})$$

are real-valued functions of system parameters, expressed in terms of digamma functions, Q_{20} is arbitrary parameter which determines the starting point in time parametrization of heteroclinic orbits in the unperturbed system.

For orbit B:

$$M^B = -\frac{\pi\sqrt{P_{20}} \sin \tilde{Q}_1}{e_A} m^B \cos[Q_{20} - \phi^B]. \qquad (7)$$

where complexes

$$m^A = m^A(f_1, f_2, \sigma, P_{20})$$
$$\phi^A = \phi^A(f_1, f_2, \sigma, P_{20})$$

are real-valued, expressed in terms of hyperbolic functions.

For orbits A', A'', B', B'' we get

$$M^{A'} = M^{A''} = M^{B'} = M^{B''} = 0.$$

According to the Melnikov theory non-traverse intersections of invariant manifolds, leading to Smale's horseshoes appear when Melnikov function has simple zeros as a function of parameters. For trivial zeros of Melnikov function we conclude that orbits A', A'', B' and B', do not break under perturbation. In cases (6,7) Melnikov function has form:

$$M^{A,B} \propto m^{A,B} \sin(Q_{20} + \phi^{A,B})$$

and as function of Q_{20} definitely has simple zeros even if all system parameters are fixed. It means that for all parameters, for which unperturbed system (4) has heteroclinic orbits, it is possible to observe chaotic behavior.

5. Numerical examples

Numerical simulation was done in order to illustrate this phenomenon. The existence of Smale horseshoes could be approved by building a sequence of Poincaré maps. We built this maps for Poincaré section given by

$$\Sigma_P = \{Q_2 = 0, Q'_2 > 0\}.$$

Results are shown in Fig. 3. They demonstrate the character of chaotical phenomena and may give us insight to physical meaning of obtained result. We see that mixed mode solutions, corresponding to fixed point (5a) and solutions that are close to it are not affected by the perturbation and demonstrate structural stability. At the same time the vicinity of unperturbed homoclinic orbits is occupied by chaotic area which arose from breaking of this orbits. The fact that orbits A', A'', B' and B' which correspond to one-mode oscillations of the plate are not destroyed by perturbation leads to conclusion that one-mode solutions are still steady state solutions of considered system, but structurally unstable.

To illustrate the role of separatrix line $P_{20} = |\beta/2|$ we give the example of Poincaré maps for parameters which lay near the boundary between regions III and IV (Fig. 4). We see that as internal detuning parameter β approaches critical value $2P_{20}$, heteroclinic loops shrink and so does chaotic region, caused by breaking of this orbits (see Fig. 4a). Finally for $\beta \geq 2P_{20}$ heteroclinic orbits disappear and considered route to chaos doesn't take place (see Fig. 4b).

Besides, computational experiments show that chaotic region is not bounded by the regions II and III. In spite of the fact that in regions I and IV unperturbed system has no heteroclinic orbits, stochastic layers on Poincaré sections built for perturbed system may appear as shown on Fig. 5. But the origin of this chaotic phenomenon is different to one described in this paper. In this case heteroclinic orbits arise from resonant KAM-tori due to perturbation according to Kolmogorov-Arnold-Moser theory [12,14] and breaking of this orbits leads to chaos. Analytical estimation of region in parametrical space where this phenomenon takes place requires further work.

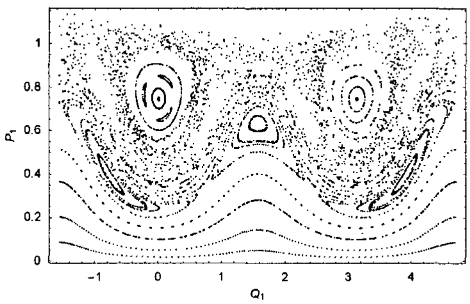
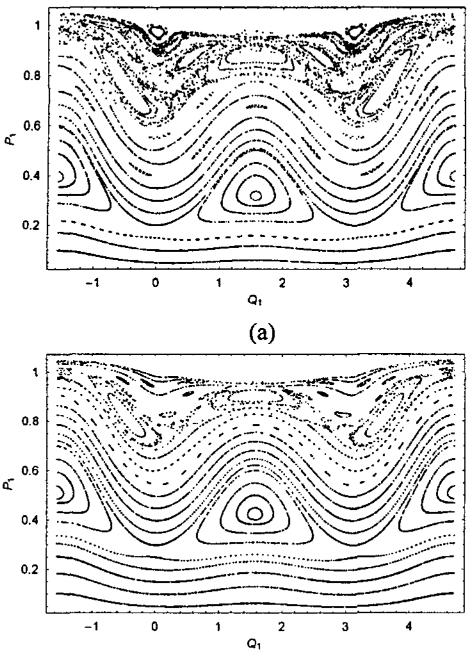


Fig. 3. Poincaré section of flow (13) for region (III). ($\varepsilon = 0.05, f_1 = 0.5, f_2 = 1.0, \sigma = -0.25, \beta = 1.0, P_{20} = 1.0$)



(a)

(b)

Fig. 4. Poincaré section of flow (13) for regions (III) and (IV).

$(a): \varepsilon = 0.05, f_1 = 0.5, f_2 = 1, \sigma = -0.25, \beta = 1.9, P_{20} = 1$

$(b): \varepsilon = 0.05, f_1 = 0.5, f_2 = 1, \sigma = -0.25, \beta = 2.1, P_{20} = 1$

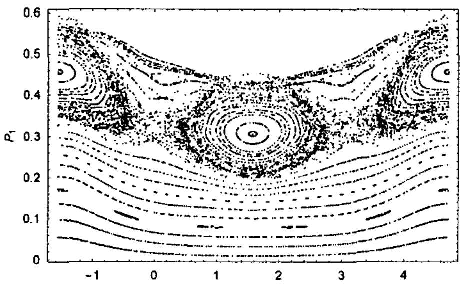


Fig. 5. Poincaré section of flow (13) for region (IV). $(\varepsilon = 0.05, f_1 = 0.5, f_2 = 0.5, \sigma = 0, \beta = 1.5, P_{20} = 0.5)$

References

[1] S. Sridhar, D.T. Mook, A.H. Nayfeh, "Nonlinear resonances in forced responses of plates, Part I: symmetric responses of circular plates", *Journal of Sound and Vibration*, 1975, 41, 359-373.

[2] X. L. Yang, P.R. Sethna, "Local and Global Bifurcations in the Parametrically Excited Vibrations of Nearly Square Plate", *International Journal of Non-linear Mechanics*, (1991) Vol. 26, 199-220.

[3] W. K. Lee and C. H. Kim, "Combination resonances of a circular plate with three-mode interaction", *American Society of Mechanical Engineers Journal of Applied Mechanics*, 1995, 62, 1015-1022.

[4] M. H. Yeo and W. K. Lee, "Corrected solvability conditions for non-linear asymmetric vibrations of a circular plate", *Journal of Sound and Vibration*, 2002, 257, 653-665.

[5] W. K. Lee and M. H. Yeo, "Nonlinear interactions in asymmetric vibrations of a circular plate", *Journal of Sound and Vibration*, 2003, 263, 1017-1030.

[6] W. K. Lee, M. H. Yeo, S. B. Samoilenko "The effect of the number of the nodal diameters on non-linear interactions in two asymmetric vibration modes of a circular plate", *Journal of Sound and Vibration*, 2003, 268, 1013-1023.

[7] Z.C. Feng and P.R. Sethna,"Global bifurcation and chaos in parametrically forced systems with one-one resonance", *Dynamics and Stability of Systems*, 1990, Vol. 5, NO. 4, 201-225.

[8] M. H. Yeo and W. K. Lee "Global bifurcations in modal interactions of a subharmonically excited circular plate"

[9] G. Kovačič and S. Wiggins, "Orbits homoclinic to resonances, with an application to chaos in a model of a forced and damped sine-Gordon equation", Physica D 57, (1992), 185-225.

[10] J. Guckenheimer, "Nonlinear Oscillations, Dynamical Systems and Bifuractions of Vector Fields",1983, Springer-Verlag N-Y, Inc.

[11] S. Wiggins, "Global Bifurcations and Chaos", 1988, Springer-Verlag N-Y, Inc.

[12] G. J. Efstathiades, "A new approach to the large-deflection vibrations of imperfect circular disks using Galerkin's procedure", *Journal of Sound and Vibration*, 1971 16, 231-253.

[13] A.H. Nayfeh, "Nonlinear oscillations",1979, John Willey & Sons, Inc.

[14] E. Ott, "Chaos in Dynamical Systems", 1993, Cambridge University Press.