Nonparametric Tests for Detecting Greater Residual Life Times

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Abstract

A nonparametric procedure is proposed to test the exponentiality against the hypothesis that one life distribution has a greater residual life times than the other life distribution. Such a hypothesis turns out to be equivalent to the one that one failure rate is greater than the other and so the proposed test works as a competitor to more IFR tests by Kochar (1979, 1981) and Cheng (1985). Our test statistic utilizes the U-statistics theory and a large sample nonpara metric test is established. The power of the proposed test is discussed by calculating the Pitman asymptotic relative efficiencies against several alter native hypotheses. A numerical example is presented to exemplify the proposed test.

Key words : Exponential distribution ; residual life times ; U-statistics ; asymptotic normality ; failure rate ; asymptotic relative efficiency.

1. Introduction

In reliability theory the concept of ageing plays a fundamental role of class- ifying the life distributions. The classes of increasing failure rate(IFR) and increasing failure rate average(IFRA) are based on the monotonicity pattern of failure rate of the distribution and the classes of decreasing mean residual life (DMRL) and new better than used in expectation(NBUE) are classified by the pattern of its mean residual life. The class of new better than used(NBU) is defined by utilizing the stochastic ordering of the residual life length. The dual classes of DFR, DFRA, IMRL, NWUE and NWU are defined by reversing the pattern of failure rate, mean residual life and residual life length. The border distribution for all of the above classes is the exponential distribution. To test the null hypothesis that the distribution is exponential against the alternative that the distribution belongs to one of the above classes, many authors have proposed a number of nonparametric tests in the literature. Proschan and Pyke (1967), Barlow and Proschan(1969), Ahmad(1975) and Ahmad(2001) propose the IFR tests. For the NBU and NBUE alternatives, there exist the nonparametric tests by Hollander and Proschan(1972), Koul(1977), Hollander and Proschan (1975) and Ahmad(2001), among many others. Hollander and Proschan(1975) and Aly(1990) also propose a class of tests against the DMRL alternatives.

Since Chikkagoudar and Schuster(1974) consider the problem of comparing two populations in terms of their failure rates, Kochar(1979, 1981) and Cheng (1985) develop several nonparametric procedures for testing the null hypothesis that two life distributions are equal against the alternative that one failure rate dominates the other. Such tests are proved to be useful to compare two used items with different life distributions in terms of their degradation process -es as they age. There exist other tests to compare two life distributions with respect to their NBU-ness or IFRA-ness by Hollander, Park and Proschan (1986) and Tiwari and Zalkikar (1991).

Let $X \geq 0$ be a life length random variable with continuous survival function $\overline{F}(x) = P(X > x)$. Then the residual life time at age t, denoted by X_t, is a random variable with continuous survival function $\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t), x, t \geq 0$. In this paper we propose a new nonparametric procedure to test the equality of two life distributions against the alternative that one life distribution has a greater residual life length than the other life distribution. The following situation illustrates how our proposed test might prove useful. Suppose that there are two groups of patients suffering from a certain type of cancer, one group being treated by a certain medical treatment and the other group being a placebo group. To check if the medical treatment is effective in treating that particular type of cancer, the medical authority wishes to test the hypothesis that the residual life length of the treatment group is stochastically greater than the residual life length of the placebo group after the treatment group receives the treatment for a certain length of time, while the placebo group is not treated at all for the same length of time. If such a hypothesis is accepted, then the treatment is verified to be effective and the treatment group may have a longer residual life than the placebo group. To the best of our knowledge, no other tests have been proposed to test the stochastic ordering of two residual life lengths. However, it can be shown that the stochastic ordering of residual life lengths is equivalent to the failure rate ordering. Thus, the efficiency of our proposed test can be evaluated by calculating the Pitman asymptotic relative efficiencies with respect to the more IFR tests by Kochar (1979, 1981) and Cheng (1985).

In Section 2, we derive the test statistics for detecting greater residual life length property. Section 3 proposes a nonparametric test procedure by proving asymptotic normality of the proposed test statistic. The consistency of the test is also proved. Sections 4 presents the Pitman asymptotic relative efficiency results.

2. Test statistic for residual life length

Let X and Y be random nonnegative lifetimes of two systems with continuous distribution functions $F(t)$ and $G(t)$ and let X_t and Y_t be its corresponding residual life times (RLT) at age t, respectively. In this section we develop a test statistic for testing the null hypothesis

$H_0 : F=G$ (the common distribution is unspecified)
versus the alternative hypothesis

$H_a : X_t \overset{st}{\leq} Y_t$ for all $t \geq 0$, with strict inequality holding for some t,

based on two random samples $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ taken from F and G, respectively. We assume that $\underline{X} = (X_1,\ldots,X_m)$ and $\underline{Y} = (Y_1,\ldots,Y_n)$ are inde-pendent. H_a states that the residual life time at age t is stochastically greater when the underlying distribution is F than when the underlying distribution is G.

Note that under H_a, $\overline{F}(x+t)\overline{G}(t) \leq \overline{G}(x+t)\overline{F}(t)$ for all $t \geq 0$. Our test statistic is derived based on the following parameter

$$\Delta(F,G) \overset{def}{=} 2\int_0^\infty\int_0^\infty \{\overline{G}(x+t)\overline{F}(t) - \overline{F}(x+t)\overline{G}(t)\} dG(x+t)dF(t)$$

$$= \int_0^\infty \overline{G}^2(t)\overline{F}(t)dF(t) - 2\int_0^\infty\int_t^\infty \overline{F}(u)\overline{G}(t)dG(u)dF(t) .$$

Under H_0, $\Delta(F,G)=0$ and under H_a, $\Delta(F,G) > 0$. Thus $\Delta(F,G)$ can be used as a measure of departure from H_0 and the larger value of $\Delta(F,G)$ indicates that Y_t is stochastically greater than X_t.

Let $F_m(\overline{F}_m)$ and $G_n(\overline{G}_n)$ denote the empirical (survival) functions formed by random samples $\underline{X}$ and $\underline{Y}$, respectively. A natural nonparametric test statistic for testing H_0 versus H_a can be formed by substituting F_m and G_n in place of F and G of $\Delta(F,G)$, respectively , as

$$\begin{aligned}
\Delta_{m,n} &= \Delta(F_m, G_n) \\
&= \int_0^{\infty} \overline{G}_n^2(t)\overline{F}_m(t)dF_m(t) - 2\int_0^{\infty}\int_0^{\infty} I(u>t)\overline{F}_m(u)\overline{G}_n(t)dG_n(u)dF_m(t) \\
&= \frac{1}{m^2 n^2}\{\sum_{i_1}\sum_{i_2}\sum_{j_1}\sum_{j_2}[I(Y_{j_1} > X_{i_1})I(Y_{j_2} > X_{i_1})I(X_{i_2} > X_{i_1}) \\
&\qquad - 2I(Y_{j_1} > X_{i_1})I(X_{i_2} > Y_{j_1})\, I(Y_{j_2} > X_{i_1})]\} \\
&= \frac{1}{m^2 n^2}\sum_{i_1}\sum_{i_2}\sum_{j_1}\sum_{j_2}\phi(X_{i_1}, X_{i_2}, Y_{j_1}, Y_{j_2}),
\end{aligned}$$

where $I(a > b) = 1$ *if* $a > b$, $= 0$ *if* $a \le b$ and

$\phi(X_1, X_2, Y_1, Y_2) = I(Y_1 > X_1)I(Y_2 > X_1)I(X_2 > X_1) - 2I(Y_1 > X_1)I(X_2 > Y_1)I(Y_2 > X_1)$.

Note that $\Delta_{m,n}$ is a U-statistic utilizing the symmetric kernel for an estimable parameter $\phi(X_1, X_2, Y_1, Y_2)$. Significantly large values of $\Delta_{m,n}$ may indicate that Y has a stochastically greater residual life at age t than X. Thus, our test is to reject H_0 in favor of H_a if $\Delta_{m,n}$ is too large. In the following section we establish the asymptotic normality of $\Delta_{m,n}$ and propose the two-sample residual life test.

3. Two-sample RLT test

The limiting distribution of $\Delta_{m,n}$ can be established by applying the Hoeffding's (1948) U-statistics theory. To evaluate the asymptotic variance of $\Delta_{m,n}$, we need the following conditional expectations. Direct calculations yield

$$E\{\phi(X_1,X_2,Y_1,Y_2) \mid X_1\} = \overline{G}^2(X_1)\overline{F}(X_1) - 2\overline{G}(X_1)\int_{X_1}^{\infty}[\overline{G}(X_1) - \overline{G}(u)]dF(u),$$

$$E\{\phi(X_1,X_2,Y_1,Y_2) \mid X_2\} = \int_0^{X_2}\overline{G}^2(u)dF(u) - 2\int_0^{X_2}\overline{G}(u)[\overline{G}(u) - \overline{G}(X_2)]dF(u),$$

$$E\{\phi(X_1,X_2,Y_1,Y_2) \mid Y_1\} = \int_0^{Y_1}\overline{G}(u)\overline{F}(u)dF(u) - 2\overline{F}(Y_1)\int_0^{Y_1}\overline{G}(u)dF(u),$$

$$E\{\phi(X_1,X_2,Y_1,Y_2) \mid Y_2) \quad = \int_0^{Y_2}\overline{G}(u)\overline{F}(u)dF(u) - 2[\int_0^{Y_2}\overline{F}(u)F(u)dG(u) + \int_{Y_2}^{\infty}\overline{F}(u)F(Y_2)dG(u).$$

Thus, under $H_0: F = G$, the asymptotic null variance is obtained as

$$\begin{aligned}
\sigma_0^2 &= Var[\sum_{k=1}^{2} E\{\phi(X_1,X_2,Y_1,Y_2) \mid X_k\}] + Var[\sum_{k=1}^{2} E\{\phi(X_1,X_2,Y_1,Y_2) \mid Y_k\}] \\
&= Var\{\frac{1}{3} - \overline{F}(X_1) + \frac{2}{3}\overline{F}^3(X_1)\}^2 + Var\{\frac{1}{3} - \overline{F}(Y_1) + \frac{2}{3}\overline{F}^3(Y_1)\}^2
\end{aligned}$$

$$= \frac{2}{105} + \frac{2}{105} = \frac{4}{105} .$$

The asymptotic distribution of $\Delta_{m,n}$ can be readily obtained by direct applica -tion of Hoeffding's (1948) two-sample U-statistic theory (cf. Randles and Wolfe(1979)). The asymptotic null distribution of $\Delta_{m,n}$ is summarized as

Theorem 3.1. Let $\lambda = \lim_{n\to\infty} \frac{m}{N}$ for $0<\lambda<1$. Then ,the limiting distribution of $N^{1/2}(\Delta_{m,n} - \Delta(F,G))$ is normal with mean 0 and a finite variance as $N\to\infty$. Under H_0, the limiting distribution of $(105N)^{1/2}\Delta_{m,n}/2$ is a standard normal distribution.

Due to Theorem 3.1, the approximate α -level test is to reject H_0 in favor of H_a if $\frac{(105N)^{1/2}\Delta_{m,n}}{2} \ge z_\alpha$, where z_α is the upper α - percentile point of the standard normal distribution. This is referred to as a two-sample RLT test. Assuming that F and G are continuous, $\Delta_{m,n}$ is strictly greater than 0 under H_a and hence, the asymptotic normality $\Delta_{m,n}$ ensures the consistency of the two-sample RLT test against the class of (F,G) pairs for which H_a holds.

The asymptotic unbiasedness of the two-sample RLT test can be proved by showing that

$$\Pr\left(\frac{(105N)^{1/2}\Delta_{m,n}}{2} \ge z_\alpha\right) \ge \alpha$$

for sufficiently large N, whenever the alternative hypothesis holds true. We first observe that

$$\Pr\left(\frac{(105N)^{1/2}\Delta_{m,n}}{2} \ge z_\alpha\right) = \Pr\left(N^{1/2}(\Delta_{m,n} - \Delta(F,G) \ge z_\alpha \left(\frac{4}{105}\right)^{1/2} - N^{1/2}\Delta(F,G)\right).$$

If $\Delta(F,G)=0$, the probability is equal to α. Under H_a, $\Delta(F,G)>0$ and thus, by the asymptotic normality of $\Delta_{m,n}$ the result follows for sufficiently large N.

4. Asymptotic relative efficiency comparison

Although there have been no other tests proposed to test H_0 versus H_a in the literature, Shaked and Shantikumar(1994) have shown that the stochastic ordering of RLT is equivalent to the IFR ordering and we restate such results in the following theorem for completeness of our discussion.

Theorem 4.1. (cf. Shaked and Shantikumar(1994)) Let X_t and $X_{t'}$ be the RLT's at ages t and t', respectively, when the underlying life distribution is F. Then, the following relations hold.

1) F belongs to the IFR class if and only if $X_t \overset{st.}{\geq} X_{t'}$ for all $t' \geq t \geq 0$.

2) F belongs to the NBU class if and only if $X_0 \geq X_t$ for all $t \geq 0$.

Note that F belongs to the IFR class if $r_F(t)$, the failure rate of F, is increasing in $t \geq 0$ and F is NBU if $\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y)$ for all x, $y \geq 0$. It is easy to check that

Theorem 4.2. Let $r_F(t)$ and $r_G(t)$ be the failure rate of the life distributions F and G, respectively. Then, the following statements are equivalent.

1) $r_F(t) \geq r_G(t)$ for all $t \geq 0$.
2) $\overline{F}(t)/\overline{G}(t)$ is decreasing in $t \geq 0$.
3) $X_t \overset{st.}{\leq} Y_t$ for all $t \geq 0$.

Due to Theorem 4.2, the hypotheses H_0 and H_a are equivalent to the hypotheses $H_0: F = G$ against $H_a': r_F(t) \geq r_G(t)$ for all $t \geq 0$. Thus, the more IFR tests of H_0 versus H_a' proposed by Kochar(1979, 1981) and Cheng (1985) can be considered as competitors to the two-sample RLT test. Among those tests, Cheng's(1985) test is known to have the best overall performance under several alternatives. Hence, the two-sample RLT test is compared with Cheng's(1985) test by computing the Pitman asymptotic relative efficiencies under the following three alternatives :

$$H_1: \overline{F}(t) = [\overline{G}(t)]^{1+\theta} \quad (\theta > 0),$$

$$H_2: \overline{F}(t) = (1-\theta)\overline{G}(t) + \theta\overline{G}(t)[1-G^k(t)] \quad (\theta > 0, k > 1/2),$$

$$H_3: \overline{F}(t) = \overline{G}(t)\exp[-\frac{1}{2}\theta\{\ln\overline{G}(t)\}^2] \quad (\theta > 0).$$

Cheng's(1985) test (henceforth, referred to as T-test) utilizes the rank statistics and its test statistic is defined as

$$T = T_{1N} / T_{2N} ,$$

where

$$T_{1N}=\sum_{i=1}^{m-1}\{\frac{n+i-R_{(i)}}{(2m^2n^2+m^2ni-mn^2i-m^2nR_{(i)})^{1/2}}\}+\frac{N^{3/2}-N^{1/2}R_{(m)}}{(m^2n^2+m^2nN^2-m^2nNR_{(m)})^{1/2}},$$

$$T_{2N}=\sum_{j=1}^{n-1}\{\frac{m+j-S_{(j)}}{(2m^2n^2+mn^2j-m^2nj-nm^2S_{(j)})^{1/2}}\}+\frac{N^{3/2}-N^{1/2}S_{(n)}}{(m^2n^2+mn^2N^2-mn^2NS_{(n)})^{1/2}},$$

$N=m+n$, and $R_{(1)}<\cdots<R_{(m)}$ and $S_{(1)}<\cdots<S_{(n)}$ are the ordered ranks of $X's$ and $Y's$ in the combined sample.

Let $\lambda=\lim_{N\to\infty}\frac{m}{N}$. Then, direct calculations yield

$$ARE_{H_1}(\Delta_{m,n},T)=\frac{105}{227.84\lambda(1-\lambda)} \quad \text{for } H_1,$$

$$ARE_{H_2}(\Delta_{m,n},T)=\frac{105}{18\lambda(1-\lambda)}[\frac{\delta(k)}{\xi(k)}]^2 \quad \text{for } H_2,$$

$$ARE_{H_3}(\Delta_{m,n},T)=\frac{105}{655.36\lambda(1-\lambda)} \quad \text{for } H_3,$$

where $\delta(k)=\frac{1}{(k+1)(k+2)}+\frac{2}{k+2}B(2,k+3)-\frac{2}{k+1}B(2,k+2)-B(4,k+1)$,

$\xi(k)=\Gamma(\frac{3}{2})\Gamma(k+1)\{\Gamma(\frac{5}{2}+k)^{-1}$ and $B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is a beta function.

Table 4.1. Pitman asymptotic relative efficiencies of the twp-sample RLT test with respect to Cheng's (1985) T test

Alternatives \ λ		0.2	0.4	0.5	0.7	0.9
H1		2.880	1.920	1.843	2.195	5.222
H2	k=1	2.279	1.519	1.458	1.736	4.051
	k=2	3.925	2.617	2.512	2.991	6.978
	k=3	4.506	3.004	2.884	3.433	8.011
	k=4	4.581	3.054	2.932	3.490	8.144
H3		1.001	0.668	0.641	0.763	1.780

Table 4.1 gives the Pitman asymptotic relative efficiencies of the two-sample RLT test based on $\Delta_{m,n}$ with respect to Cheng's (1985) test based on T for several choice of λ and k

under the alternatives H_1 ,H_2 and H_3 . Note that the asymptotic null variance is $\frac{4}{105}$, independent of the unspecified common distribution and so the Pitman asymptotic relative efficiencies are dependent on the value of λ for all three alternatives. Table 3 clearly shows that the two-sample RLT test is superior to Cheng's(1985) test under H_1 and H_2. Under H_3, two tests seem to be competitive.

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