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퍼지수치 확률변수의 쇼케이 기댓값과 그 응용

Choquet expected values of fuzzy number-valued random variables and their applications

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Abstract

In this paper, we consider interval number-valued random variables and fuzzy number-valued random variables and discuss Choquet integrals of them. Using these properties, we define the Choquet expected value of fuzzy number-valued random variables which is a natural generalization of the Lebesgue expected value of Lebesgue expected value of fuzzy random variables. Furthermore, we discuss some application of them.

Keywords : fuzzy measures, Choquet integrals, Choquet expected value, fuzzy number-valued random variable.

1. Introduction.

It is well-known that closed set-valued functions had been used repeatedly in [1, 2, 3]. Aumann first defined the concept of integrals of set-valued functions(simply, Aumann's intgerals) with respect to a classical measure and the theory about Aumann's integrals has drawn much attention as its repeat applications in mathematics economics, conditional expectation of multi-valued functions and many other fields. But all these are based on Lebesgue integrals with respect to a classical measure. Using these ideas, we have studied closed set-valued Choquet integrals and convergence theorems under some sufficient conditions in [4,5]. And we also have tried modifying these conditions in [6,7,8,9].

In this paper, we consider interval number-valued random variables which are interval number-valued measurable functions and fuzzy number-valued random variables which were first introduced by Kwakernaak and discuss Choquet integrals of interval number-valued random variables and fuzzy number-valued random variables. We note that study of fuzzy number-valued random variables is interesting subject from point of view of both mathematical theory and its application.

Using these properties, we define the Choquet expected value of fuzzy number-valued random variables which is a natural generalization of the Lebesgue expected value of the Lebesgue expected value of Lebesgue expected value of fuzzy random variables. Furthermore, we discuss some application of them.

2. Definitions and preliminaries.

A fuzzy measure on a measurable space $(\Omega, \mathfrak{I})$ is an extended real-valued function $\mu : \mathfrak{I} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset)=0$,

(ii) $\mu(A) \le \mu(B)$, whenever $A, B \in \mathfrak{I}$, $A \subset B$.

A fuzzy measure μ is said to be autocontinuous from above[resp.,below] if $\mu(A \cup B_n) \to \mu(A)$[resp., $\mu(A \sim B_n) \to \mu(A)$] whenever $A \in \mathfrak{I}$, $\{B_n\} \subset \mathfrak{I}$ and $\mu(B_n) \to 0$. If μ is autocontinuous both from above and from below, it is said to be autocontinuous(see [12]). We note that a random variable is a measurable function $y: \Omega \to [0, \infty)$.

Definition 2.1 ([10, 14]) Let μ be a fuzzy measure on $(\Omega, \mathfrak{I})$ and y a random variable.

(1) The Choquet integral of a measurable function y with respect to a fuzzy measure μ is defined by

$$(C)\int y d\mu = \int_0^\infty \mu(\{w \in \Omega | y(w) > r\}) dr$$

where the integral on the right-hand side is an ordinary one.

(2) If Ω is a finite set $\Omega = \{w_1, \cdots, w_n\}$, the Choquet integral of y can be written as follows:

$$(C)\int y d\mu = \sum_{i=1}^{n} (y(w_{(i)}) - y(w_{(i-1)}))\mu(A_{(i)})$$

where the subscript $(\cdot)$ indicates that the indices have been permited in order to have

$$y(w_{(1)}) \le y(w_{(2)}) \le \cdots \le y(w_{(n)}),$$

$A_{(i)} = \{w_{(i)} \cdots w_{(n)}\}$

and $y(w_{(0)}) = 0$, by convention.

(3) A random variable y is called Choquet integrable if the Choquet integral of f can be defined and its value is finite.

Throughout the paper, R^+ will denote the interval $[0, \infty)$,

$$I(R^+) = \{[a, b] | a, b \in R^+ \text{ and } a \leq b\}.$$

Then a element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$[a, b] + [c, d] = [a+c, b+d]$,
$[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$,
$k[a, b] = [ka, kb]$,
$[a, b] \sqsubseteq [c, d]$ if and only if $a \leq c$ and $b \leq d$,
$[a, b] \sqsubset [c, d]$ if and only if
$[a, b] \sqsubseteq [c, d]$ and $[a, b] \neq [c, d]$
$[a, b] < [c, d]$ if and only if either
$a < c$ or $(a = c$ and $b < d)$,
$[a, b] \leq [c, d]$ if and only if
$[a, b] < [c, d]$ or $[a, b] = [c, d]$,
$[a, b] \subset [c, d]$ if and only if
$[a, b]$ is a subset of $[c, d]$.

We note that $(I(R^+), d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A, B) = \max\{ \sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y| \}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

A fuzzy number V is a fuzzy set on R^+, satisfying the following conditions([9,10,11,13,16]):

(i) (normality) $V(x) = 1$ for some $x \in R^+$,

(ii) (fuzzy convexity) for every $\lambda \in (0, 1]$, $[V]^\lambda = \{x \in R^+ | V(x) \geq \lambda\} \in I(R^+)$,

(iii) $[V]^0 = cl\{x \in R^+ | V(x) > 0\} \in I(R^+)$, where clA is the closure of A in the usual topology of R^+.

Let $F(R^+)$ denote the class of fuzzy numbers. We define; for each pair $V, W \in F(R^+)$,

$[V + W]^\lambda = [V]^\lambda + [W]^\lambda$, for all $\lambda \in [0, 1]$
$[VW]^\lambda = [V]^\lambda [W]^\lambda$ for all $\lambda \in [0, 1]$
$[kV]^\lambda = k[V]^\lambda$, for all $\lambda \in [0, 1]$
$V \leq W$ if and only if $[V]^\lambda \leq [W]^\lambda$ for all $\lambda \in [0, 1]$
$V \sqsubseteq W$ if and only if $[V]^\lambda \sqsubseteq [W]^\lambda$, for all $\lambda \in [0, 1]$
$V \sqsubset W$ if and only if $[V]^\lambda \sqsubset [W]^\lambda$, for all $\lambda \in [0, 1]$
$V < W$ if and only if $[V]^\lambda < [W]^\lambda$, for all $\lambda \in [0, 1]$
$V \leq W$ if and only if $[V]^\lambda \leq [W]^\lambda$, for all $\lambda \in [0, 1]$
$V \subset W$ if and only if $[V]^\lambda \subset [W]^\lambda$, for all $\lambda \in [0, 1]$.

We also note that $(F(R^+), D)$ is a metric space, where $D : F(R^+) \times F(R^+) \to [0, \infty]$ is defined by

$$D(V, W) = \sup\{ d_H([V]^\lambda, [W]^\lambda) | \lambda \in (0, 1] \}.$$

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $Y : \Omega \to C(R^+) \setminus \{\emptyset\}$ and an interval number-valued function $Y : \Omega \to I(R^+) \setminus \{\emptyset\}$. We denote that $d_H - \lim_{n \to \infty} A_n = A$ if and only if $\lim_{n \to \infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$. We say that a closed set-valued function Y is a closed set-valued random variable if it is measurable, that is, for each open set $O \subset R^+$,

$$Y^{-1}(O) = \{w \in \Omega | Y(w) \cap O \neq \emptyset\} \in \mathfrak{I}.$$

Definition 2.2 ([1,2,3]) Let Y be a closed set-valued random variable. A random variable $y : \Omega \to R^+$ satisfying

$$y(w) \in Y(w) \text{ for all } w \in \Omega$$

is called a random selection of Y.

We say $y : \Omega \to R^+$ is in $L_c^1(\mu)$ if and only if y is random variable and $(C) \int y \, d\mu < \infty$. We note that " $w \in \Omega$ $\mu - a.e.$" stands for " $w \in \Omega$ μ-almost everywhere". The property $P(w)$ holds for $w \in \Omega$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $P(w)$ holds for all $w \in A^c$, where A^c is the complement of A.

Definition 2.3 [8,9,17] Let Y be a closed set-valued random variable and $A \in \mathfrak{I}$.

(1) The Choquet expected value $E_c(Y)$ of Y on A is defined by

$$E_c(Y) \equiv (C) \int_A Y d\mu = \{(C) \int_A y d\mu \mid y \in S(Y)\}$$

where $S(Y)$ is the family of $\mu - a.e.$ random selections of Y, that is,

$$S(Y) = \{y \in L_c^1(\mu) \mid y(w) \in Y(w) \ w \in \Omega \ \mu - a.e.\}.$$

(2) A closed set-valued random variable Y is said to be Choquet integral existing if $(C) \int Y d\mu \neq \emptyset$, and it is said to be Choquet integrable if $(C) \int Y d\mu$ exists and does not include ∞.

(3) A closed set-valued random variable Y is said to be Choquet integrably bounded if there is a function $g \in L_c^1(\mu)$ such that

$$\| Y(w) \| = \sup_{r \in Y(w)} |r| \leq g(w) \quad \text{for all } w \in \Omega.$$

We remark that if Y is Choquet integrable, then we have $S(Y) \subset L_c^1(\mu)$. This implies that Definition 2.3

is equal to the definition of closed set-valued Choquet integrals in [4,5,6,7,8,9].

Theorem 2.4 (1) If $Y:\Omega\to I(R^+)$ is an interval number-valued random variable and if we define

$$y^\star(w)=\sup\{r|r\in Y(w)\}$$

and

$$y_\star(w)=\inf\{r|r\in Y(w)\}$$

for all $w\in\Omega$, then $y^\star$ and $y_\star$ are random variables.
(2) If $Y:\Omega\to I(R^+)$ is a Choquet integrably bounded interval number-valued random variable, then then $y^\star$ and $y_\star$ are random selections and $E_c(Y)$ is bounded and nonempty set.

Theorem 2.5 Let μ be a continuous fuzzy measure. If $Y:\Omega\to I(R^+)$ is a Choquet integrably bounded interval number-valued random variable, then $E_c(Y)\in I(R^+)$, that is, $E_c(Y)=[(C)\int y^\star d\mu,\ (C)\int y_\star d\mu]$.

3. The Choquet expected value of fuzzy number-valued random variables.

Definition 3.1 (1) A mapping $\tilde{Y}:\Omega\to F(R^+)$ is called a fuzzy number-valued random variable if for each $\alpha\in[0,1]$, $[\tilde{Y}]^\alpha:\Omega\to I(R^+)$ is an interval number-valued random variable.

(2) $\tilde{Y}$ is called Choquet integrably bounded if $[\tilde{Y}]^0$ is Choquet integrably bounded.

Let $\tilde{Y}:\Omega\to F(R^+)$ be Choquet integrably bounded fuzzy number-valued random variable. We define the Choquet expected value(denoted by $E_c(\tilde{Y})$) of $\tilde{Y}$ as that element $V\in F(R^+)$ which satisfies

$$[V]^\lambda=(C)\int[\tilde{Y}]^\lambda d\mu,\ \text{for all } \lambda\in[0,1].$$

We should prove that effectively the family $\{(C)\int [\tilde{Y}]^\lambda d\mu \mid \lambda\in[0,1]\}$ defines a fuzzy set for this purpose, we using the following lemma:

Lemmma 3.2 ([13], Lemma 2.1) If $\{[a^\lambda,b^\lambda]\mid\lambda\in[0,1]\}$ is a given family of nonempty interval numbers. If (i) for all $0\le\lambda_1\le\lambda_2\le1$, $[a^{\lambda_1},b^{\lambda_1}]\supset[a^{\lambda_2},b^{\lambda_2}]$ and (ii) for any nonincreasing sequence $\{\lambda_k\}$ in $[0,1]$ in converging to λ, $[a^\lambda,b^\lambda]=\cap_{k=1}^\infty[a^{\lambda_k},b^{\lambda_k}]$. Then there exists a unique fuzzy number $V\in F(R^+)$ such that the family $[a^\lambda,b^\lambda]$ represents the λ-level sets of V.

Conversely, if $[a^\lambda,b^\lambda]$ are the the λ-level sets of a fuzzy number $V\in F(R^+)$, there the conditions (i) and (ii) are satisfied.

In order to see that $\{(C)\int [\tilde{Y}]^\lambda d\mu\mid\lambda\in[0,1]\}$ define a fuzzy number in $F(R^+)$, we check (i) and (ii) as in the following:

(i) if $0\le\lambda_1\le\lambda_2\le1$, then we have $[\tilde{Y}]^{\lambda_1}(w)\subset[\tilde{Y}]^{\lambda_2}(w)$, for all $w\in\Omega$. That is, $y\in S([\tilde{Y}]^{\lambda_2})$ implies $y\in S([\tilde{Y}]^{\lambda_1})$. Thus, we have $(C)\int[\tilde{Y}]^{\lambda_2}d\mu\subset(C)\int[\tilde{Y}]^{\lambda_1}d\mu$, and

(ii) let $\{\lambda_n\}$ with $\lambda_n\uparrow\lambda$, we have to see that

$$(C)\int[\tilde{Y}]^{\lambda_n}d\mu=\cap_{n=1}^\infty(C)\int[\tilde{Y}]^{\lambda_n}d\mu.$$

Therefore, we can obtain the following:

Theorem and Definition 3.3 Let $\tilde{Y}:\Omega\to F(R^+)$ be Choquet integrably bounded fuzzy number-valued random variable. Then there exists a uniquely fuzzy number $E_c(\tilde{Y})$ with λ-level sets $E_c([\tilde{Y}]^\lambda)$. $E_c(\tilde{Y})$ is called the Choquet expected value of $\tilde{Y}$.

4. The Choquet expected fuzzy number-valued utility function.

Expected utility theory combines linearity in probabilities and a utility function, which is either concave or convex if a dicision-maker is risk averse or seeking. However, maximization of expected utility as a criterion of choice among the alternatives involving risk fails to explain the existence of both insurance and lotteries. In the paper [14,15], given an utility function u, such that $u:\Omega\to R^+$, a fuzzy measure μ on Ω and a set X of comonotonic prospects $x:\Omega\to R^+$, such that $x,x'\in X$ are comonotonic if and only if there are no $w_1,w_2\in\Omega$ such that $x(w_1)>x(w_2)$ and $x'(w_1)<x'(w_2)$, the Choquet integral permits the evaluation of the Choquet

expected utility function as in the following theorem:

Representation Theorem 4.1 ([14]) Let u be a utility function, then for every $x, x' \in X$,

$$x \geq x' \leftrightarrow U(x) \geq U(x')$$

where U is defined as $U(x) = (C)\int u(x(\cdot))d\mu$ (Choquet integral with respect to μ).

Finally, we discuss a fuzzy number-valued utility function as in the following:

$$\tilde{u} : \Omega \rightarrow F(R^+)$$

We note that $[\tilde{u}]^\lambda$ is an interval number-valued utility function for all $\lambda \in [0,1]$. Then by Theorem 2.5, we have the Choquet expected fuzzy number-valued utility function $\tilde{U}$ as in the following:

$$[\tilde{U}]^\lambda(x) = (C)\int[\tilde{u}]^\lambda(x(\cdot))d\mu$$

$$= [(C)\int[\tilde{u}]^\lambda_*(x(\cdot))d\mu,\ (C)\int[\tilde{u}]^{\lambda *}(x(\cdot))d\mu]$$

and

$$\tilde{U}(x) = (C)\int\tilde{u}(x(\cdot))d\mu.$$

Therefore, we can obtain the following theorem for fuzzy number-valued utility functions:

Representation Theorem 4.2 Let $\tilde{u}$ be a fuzzy number-valued utility function, then for every $x, x' \in X$,

$$x \geq x' \leftrightarrow \tilde{U}(x) \sqsupseteq \tilde{U}(x')$$

where $\tilde{U}$ is defined as $\tilde{U}(x) = (C)\int \tilde{u}(x(\cdot))d\mu$

Remark 4.3 If Ω is a finite set $\Omega = \{w_1, \cdots, w_n\}$, then we have the Choquet expected fuzzy number-valued utility function as in the following:

$$\tilde{U}(x) = \sum_{i=1}^{n} [\tilde{u}(x(w_{(i)})) - \tilde{u}(x(w_{(i-1)}))]\mu(A_{(i)}).$$

References.

[1] J. Aubin, Set-valued analysis, 1990, Birkauser Boston.

[2] R. J. Aumann, Integrals of set-valued functions, J. Math. Anal. Appl. 12 (1965)1-12.

[3] F. Hiai and H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functions, J. Multi. Analysis 7(1977) 149-182.

[4] L. C. Jang, B. M. Kil, Y. K. Kim and J. S. Kwon, Some properties of Choquet integrals of set-valued functions, Fuzzy Sets and Systems 91 (1997) 95-98.

[5] L. C. Jang and J. S. Kwon, On the representation of Choquet integrals of set-valued functions and null sets, Fuzzy Sets and Systems 112(2000)233-239.

[6] L. C. Jang and T. Kim, On set-valued Choquet intgerals and convergence theorems, Advan. Stud. Contemp. Math. 6(1) (2003), 63-76.

[7] L.C. Jang, T. Kim and J.D. Jeon, On set-valued Choquet integrals and convergence theorems (II), Bull. Korean Math. Soc. 40(1) (2003), 139-147.

[8] L.C. Jang, T. Kim and J.D. Jeon, Oncomonotonically additive interval-valued functionals and interval-valued Choquet integrals (II), J. of Fuzzy Logic and Intelligent Systems 14(1) (2004), 33-38.

[9] L.C. Jang, T. Kim and J.D. Jeon, On Choquet integrals of measurable fuzzy number-valued functions, Bull. Korean Math. Soc. 41(1) (2004), 95-107.

[10] V. Kratschmer, Limit theorems for fuzzy random variables, Fuzzy Sets and Systems 126(2002), 253-263.

[11] H. Kwakernaak, Fuzzy random variables-Definition and Theorems, Inf. Science 15(1978), 1-29.

[12] T. Murofushi and M. Sugeno, A theory of Fuzzy measures: representations, the Choquet integral, and null sets, J. Math. Anal. and Appl. 159 (1991) 532-549.

[13] M.L. Puri and D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114(1986), 409-422.

[14] Richard Alo, Andre de Korvin and Francois Modave, Decision Making for Robust Resilient systems, Proceedings of the 36th Hawaii international conference on System Science-2003.

[15] D. Schmeidler, Integral representation without additivity, Proc. Amer. Math. Soc. 97(1986), 253-261.

[16] B. Zhang, On measurability of fuzzy number-valued functions, Fuzzy Sets and Systems 120(2001),505-509.

[17] D. Zhang, C.Guo and Dayou Liu, Restudy on set valued Choquet integrals, To appear in Fuzzy Sets and Systems.