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Intuitionistic Fuzzy Subgroups and Level Subgroups

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Abstract

In this paper, we introduce the concept of level subgroups of an intuitionistic fuzzy subgroup, and study some properties of level subgroups in the first part of the paper. These level subgroups in turn play an important role in the characterization of all intuitionistic fuzzy subgroups of a prime cyclic group.

Key words and phrases : intuitionistic fuzzy set, intuitionistic fuzzy subgroup, level subgroup.

0. Introduction

The concept of fuzzy sets was introduced by Zadeh in [14]. Since its inception, the theory of fuzzy sets has developed in many directions and is finding applications in a wide variety of fields. In particular, several researchers [6, 12, 13] have applied the notion of fuzzy sets to algebra.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Recently, Çoker and his colleagues [4,5,7] applied the notion of intuitionistic fuzzy sets to topology. In 1989, Biswas [3] introduced the concept of intuitionistic fuzzy subgroups and investigated some of it's properties. Moreover, Hur and his colleagues [8,9,10] redefined the concept of intuitionistic fuzzy subgroupoids, subgroups and rings, and studied some of their properties. In particular, they gave a characterization of all intuitionistic fuzzy subgroups of a prime cyclic group in tems of the complex mapping [10, Proposition 2.14].

In this paper, we obtain a similar characterization of all intuitionistic fuzzy subgroups of finite cyclic groups. For this, we study some properties of level subgroupss of an intuitionistic fuzzy subgroup in the first part of the paper. These level subgroups in turn play an important role in the above characterization.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f=(f_1,f_2):X\to Y\times Z$ is called a *complex mapping* if $f_1:X\to Y$ and $f_2:X\to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I.

Definition 1.1[2]. Let X be a nonempty set. A complex mapping $A=(\mu_A,\nu_A):X\to I\times I$ is called an *intuitionistic fuzzy set* (in short, it IFS) on X if $\mu_A+\nu_A\leq 1$. where the mapping $\mu_A:X\to I$ and $\nu_A:X\to I$ denote the degree of
membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x\in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[2]. Let X be a nonempty set and let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be IFSs on X. Then

(1) $A\subset B$ iff $\mu_A\leq\mu_B$ and $\nu_A\geq\nu_B$.
(2) $A=B$ iff $A\subset B$ and $B\subset A$.
(3) $A^c=(\nu_A,\mu_A)$.
(4) $A\cap B=(\mu_A\wedge\mu_B,\nu_A\vee\nu_B)$.
(5) $A\cup B=(\mu_A\vee\mu_B,\nu_A\wedge\nu_B)$.
(6) $[\ ]A=(\mu_A,1-\mu_A)$, $\langle\ \rangle A=(1-\nu_A,\nu_A)$.

Definition 1.3[4]. Let $\{A_i\}_{i\in J}$ be an arbitrary family of IFSs in X, where $A_i=(\mu_{A_i},\nu_{A_i})$ for each $i\in J$. Then

(a) $\bigcap A_i=(\wedge\mu_{A_i},\vee\nu_{A_i})$.
(b) $\bigcup A_i=(\vee\mu_{A_i},\wedge\nu_{A_i})$.

Definition 1.4[4]. $0_{\sim}=(0,1)$ and $1_{\sim}=(1,0)$.

Definition 1.5[9]. Let A be an IFS in a set X and let $\lambda,\mu\in I$ with $\lambda+\mu\leq 1$. Then the set $A^{(\lambda,\mu)}=x\in X:\mu_A(x)\geq\lambda$ and $\nu_A(x)\leq\mu$ is called a (λ,μ)*-level subset* of *A*.

Result 1.A[8, Proposition 2.2]. Let A be an IFS in a set X and let (λ_1,μ_1), $(\lambda_2,\mu_2)\in \mathrm{Im}(A)$. If $\lambda_1\leq\lambda_2$ and $\mu_1\geq\mu_2$, then $A^{(\lambda_2,\mu_2)}\subset A^{(\lambda_1,\mu_1)}$.

Definition 1.6[10]. Let G be a group and let $A\in IFS(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, *IFG*) of G if it satisfies the following conditions:

(i) A is an IFGP of G.
(ii) $\mu_A(x^{-1})\geq\mu_A(x)$ and $\nu_A(x^{-1})\leq\nu_A(x)$ for each $x\in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 1.B[10, Proposition 2.6]. Let A be an IFG of a group G. Then $A(x^{-1})=A(x)$ and $\mu_A(x)\leq\mu_A(e)$, $\nu_A(x)\geq\nu_A(e)$ for each

$x \in G$, where e is the identity element of G.

Result 1.C[10, Proposition 2.17 and Proposition 2.18]. Let A be an IFS of a group G. Then $A \in IFG(G)$ if and only if for each $(\lambda, \mu) \in I \times I$ with $(\lambda, \mu) \leq A(e)$, i.e., $\lambda \leq \mu_A(e)$, and $\mu \geq \nu_A(e)$, $A^{(\lambda, \mu)}$ is a subgroup of G.

Result 1.D[10, Proposition 2.14]. Let G_p be the cyclic group of prime order p. Then $A \in IFG(G_p)$ if and only if $A(x) \leq A(e)$, i.e., $\mu_A(x) = \mu_A(y) \leq \mu_A(e)$ and $\nu_A(x) = \nu_A(y) \geq \nu_A(e)$ for any $x, y \in G_p$ such that $x \neq e$ and $y \neq e$.

2. Level subgroups

Definition 2.1. Let G be a group and let $A \in IFG(G)$. Then the subgroups $A^{(\lambda, \mu)}$ are called (λ, μ)-*level subgroups* of A.

Let G be a finite group. Then the number of subgroups of G is finite. However, the number of (λ, μ)-level subgroups of an IFG A appears to be infinite. Indeed, since every (λ, μ)-level subgroup is a subgroup of G, not all these (λ, μ)-level subgroups are distinct.

Theorem 2.2. Let G be a group and let $A \in IFG(G)$. Two level subgroups $A^{(t_1, s_1)}$ and $A^{(t_2, s_2)}$ (with $(t_1, s_1) < (t_2, s_2)$, i.e., $t_1 < t_2$ and $s_1 > s_2$) of A are equal if and only if there is no $x \in G$ such that $(t_1, s_1) < A(x) < (t_2, s_2)$, i.e., $t_1 < \mu_A(x) < t_2$ and $s_2 > \nu_A(x) > s_1$.

Corollary 2.2. Let G be a finite group of order n and let $A \in IFG(G)$.. Let $\mathrm{Im}(A) = \{(t_i, s_i) : A(x) = (t_i, s_i) \text{ for some } x \in G\}$. Then $\{A^{(t_i, s_i)}\}$ are the only level subgroups of A.

Theorem 2.3. Any subgroup H of a group G can be realized as a level subgroup of some IFG of G.

The following result is the generalization of Theorem 2.3:

Theorem 2.4. Let G be a group and let the following be any chain of subgroups

$$C_0 \subset C_1 \subset \cdots \subset C_r = G.$$

Then there exists an intuitionistic fuzzy subgroup of G whose level subgroups are precisely the members of this chain.

Theorem 2.5. Let G be a group and let $LG(G) = \{A^{(\lambda, \mu)} : A^{(\lambda, \mu)}\}$ is a level subgroup of A and $A \in IFG(G)$. Let SG(G) be the set of all subgroups of G. Then there is a bijection $f: LG(G) \sim \to SG(G)$, where $\sim$ is a suitable equivalence relation on $LG(G)$.

As a consequence of Theorem 2.2, the level subgroups of an IFG A form a chain. Since $A(x) \leq A(e)$, i.e., $\mu_A(x) \leq \nu_A(e)$ and $\nu_A(x) \geq \nu_A(e)$ for each $x \in G$, $A^{(t_0, s_0)}$ is the smallest level subgroup of A, where $A(e) = (t_0, s_0)$. Thus we have the chain

$$(e) = A^{(t_0, s_0)} \subset A^{(t_1, s_1)} \subset \cdots \subset A^{(t_r, s_r)} = G \quad (*)$$

where $(t_0, s_0) > (t_1, s_1) > \cdots > (t_r, s_r)$, i.e., $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. We denote this chain $(*)$ of level subgroups by $C(A)$. In general, as all the subgroups of G do not form a chain, it follows that not all subgroups of G are level subgroups of a given fuzzy subgroup. So it is an interesting problem to find an IFG A of G which accommodates as many subgroups of G as possible in $C(A)$.

Theorem 2.6. Let G be a finite group such that $G = C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$, where the C_{p_i} are prime cyclic groups of orders p_i. Then there exists an $A \in IFG(G)$ such that $C(A)$ is a maximal chain of length $r+1$.

Remark 2.7. In the same way, we can find an *IFG* A with the maximal $C(A)$ in the following cases :

(i) G is a cyclic p-group.

(ii) G is the direct product of cyclic p-group.

(iii) G is a finite abelian group.

We can easily check these cases by adopting the same technique as proof in Theorem 2.6.

3. Characterization of intuitionistic fuzzy subgroups of finite cyclic groups.

Theorem 3.1. Let G be a cyclic p-group of order p^n, where p is a prime. Let $A \in IFG(G)$, let $x, y \in G$ and let $O(x)$ denote the order of x.

(1) If $O(x) > O(y)$, then $A(y) \geq A(x)$, i.e., $\mu_A(y) \geq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$.

(2) If $O(x) = O(y)$, then $A(x) = A(y)$.

Theorem 3.1 is not true in general as shown in the following examples:

Example 3.2. Consider the Kleins 4-group :

$$G = \{a, b: a^2 = b^2 = (ab)^2 = e\}.$$

We define a complex mapping $A = (\mu_A, \nu_A): G \to I \times I$ as follows:

$$A(e) = (t_0, s_0),\ A(a) = (t_1, s_1),$$
$$A(b) = (t_2, s_2) = A(ab),$$

where $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$ and $t_i + s_i \leq 1$ for $i = 0, 1, 2$. Then clearly $A \in IFG(G)$. But, even though $O(a) = O(b)$, $A(a) \neq A(b)$.

For a cyclic group it can be seen that $O(a) = O(b)$ implies $A(a) = A(b)$. But $O(a) \neq O(b)$ may also imply $A(a) = A(b)$.

Example 3.3. Let $G = (a)$ be a cyclic group of order 6. We define a complex mapping $A = (\mu_A, \nu_A): G \to I \times I$ as follows:

$$A(e) = (t_0, s_0), A(a) = A(a^3) = A(a^5) = (t_1, s_1),$$
$$A(a^2) = A(a^4) = (t_2, s_2),$$

where $(t_0, s_0) > (t_1, s_1) > (t_2, s_2)$ and $t_i + s_i \leq 1$ for $i = 0, 1, 2$. Then clearly $A \in IFG(G)$ and $O(a^3) \neq O(a)$. But $A(a) = A(a^3)$.

Now we give the characterization of all IFGs of a finite cyclic group in the following. In fact, the following result is the spacial case of Theorem 2.4:

Theorem 3.4. Let G be a finite cyclic group and let $A \in IFS(G)$. Then A is an *IFG* of G if and only if there exists a maximal chain of subgroups $(e) = C_0 \subset C_1 \subset \cdots \subset C_r = G$ such that forany (t_0, s_0), (t_1, s_1), $\cdots$, $(t_r, s_r) \in \mathrm{Im}(A)$ with $(t_0, s_0), > (t_1, s_1) > \cdots > (t_r, s_r)$, $A(e) = (t_0, s_0)$, $A(\mathcal{C}_1) = (t_1, s_1)$, $\cdots$, $A(\mathcal{C}_r) = (t_r, s_r)$, where $\mathcal{C}_i = C_i - C_{i-1}$ for $i = 1, 2, \cdots, r$.

The following is the immediate result of Theorem 3.4:

Corollary 3.4. Let G be a cyclic p-group of order p^r and let $A \in IFS(G)$. Then $A \in IFG(G)$ if and only if for each $x \in G$ with $O(x) = p^i$, $A(x) = (t_i, s_i)$, where $i = 0, 1, \cdots, r$ and $(t_0, s_0) > (t_1, s_1) > \cdots > (t_r, s_r)$.

Remark 3.5. We can also prove this Corollary by using Theorem 3.1.

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