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Development of higher performance algorithm for dynamic PIV

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Abstract: The new algorithm for higher performance of dynamic PIV has been proposed. Present study considered mathematical basis of PIV analysis for multiple-time-step images and it enables us to analyze the high time-resolution PIV, which is obtained by dynamic PIV system. Conventional single pair image PIV analysis gives us the velocity field data in each time step but it sometimes contains unnecessary information of target flow. Present technique utilize multi-time step correlation information, and it is analyzed

Keywords: Dynamic PIV, multi-time step, asymptotic method.

1. Introduction

Recent developments of instruments for image measurement are remarkable. It is not so difficult to obtain the multi-mega pixels or the ultra-short time interval successive images. The compact and high power pulse laser is regularly used, and conventional PIV algorithms work steady enough for ordinary applications. Those significant equipments have further abilities not only for the higher accuracy and resolution but also for the development of another dimension of measurement. For example, the successive time images with over 1KHz sampling frequency have enough time resolute for ordinary scale of flow, but the image set will have further information such as acceleration or fluctuation of smaller flow scale. Simple accumulations of conventional PIV results enable us to obtain such information by applying finite difference or FFT on the measured velocity filed. But it will have limitation in accuracy and resolution caused by the digital resolution limit, the accuracy of finite difference and the accumulation of residual error. It has been required that applicable algorithms for higher potential PIV images is to be developed, and it will open next stage of PIV measurement.

One of the authors has established the analytical basis of PIV measurement (Nishio, 2001), and it has contributed to understand the evaluation of PIV image mathematically. The analytics of PIV measurement enables us to evaluate the PIV image appropriately and it indicates how we can extend the principle of PIV analysis. In the present paper, the basis of analytics for PIV measurement are summarized, and the applications for multi-time-step PIV images were considered. Asymptotic approaches to obtain the higher order solutions were considered, and innovative algorithms were proposed.

2. Constraint equation and evaluation indexes

2.1 Basic analytics of PIV

The basic analytics of PIV is summarized in this section, and it is the basis for the extended algorithms for multi-time-step PIV. Simplifying mathematical expressions and discussions, the luminance function is standardized by Eqs.(1),(2),(3). $\hat{f}$ shows the luminance of original image and S shows the area of target space such as correlation window. $\bar{f}$ and σ^2 show the mean value and deviation of luminance inside the target space, and f shows the standardized luminance function. The standardized luminance enables us to eliminate the external effects such as the non-uniformity of illumination or the dependency of image contrast.

$$\bar{f} = \frac{1}{S}\int_S \hat{f}\, dS \tag{1}$$

$$\sigma^2 = \frac{1}{S}\int_S \left(\hat{f} - \bar{f}\right)^2 dS \tag{2}$$

$$f = \frac{\hat{f} - \bar{f}}{\sigma} \tag{3}$$

The standardization is already employed in the original principle of correlation method (Kimura, 1987), but it has not been used in the constraint equation of gradient method. It would largely contribute to the robustness of analysis for the change of image condition such as contrast or illumination. All the mathematical expressions appeared in the following sections will use the standardized luminance, but the essential of the analytical discussion does not change.

As it is described in the previous paper (Nishio, 2001), the constraint equation of image displacement is expressed by a differential equation shown by Eq.(4). Figure 1 shows a schematic view of the transport of luminance of image. The luminance is transported by the convection, and it will be changed by external factors such as non-uniform illumination field. The differential equation is obtained by applying the limitation $\Delta t \to 0$ on the finite difference between t and $t + \Delta t$. The external effects are neglected here as it can be discussed separately.

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0 \tag{4}$$

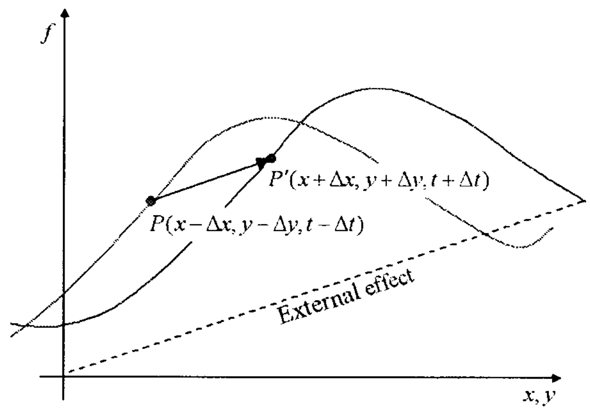

Fig.1 Schematic view of the transport of particle image luminance

The constraint equation is defined at single point in spatio-temporal space. It can be applied to the displacement when the luminance derivative is effective and that can be realized when Δt is small. However, it cannot be satisfied always, and much wider dynamic range ha been required. As Eq.(4) is satisfied everywhere in the spatio-temporal space in visualized flow field, the integral form of constraint equation can be obtained as shown by Eq.(5). s_k shows a path line that is formed in a period of $[t, t + \Delta t]$, and the result of integral can be expressed by the luminance at the both end of path line.

$$\int_{s_k} \frac{Df}{Dt} ds = (f_{k+1} - f_k) = 0 \tag{5}$$

The integral style constraint equation expresses the simple fact that the luminance does not change when we can track the particle path correctly. (Kaga, 1992) The similarity of pair image evaluated by a norm of constraint equation as shown by Eq.(6).

$$\frac{1}{S}\int_S \left\{ \int_{s_k} \frac{Df}{Dt} ds \right\}^2 dS = 0 \tag{6}$$

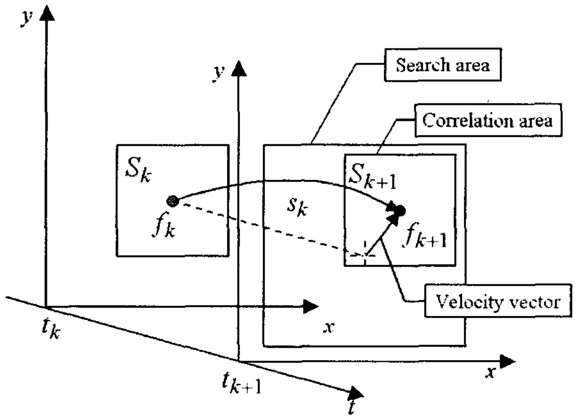

Fig.2 Schematic view of luminance transportation of PIV image.

Equation (6) shows the principle of conventional correlation method, and the relationship with the ordinary correlation coefficient is shown by Eq.(7). As the luminance function is standardized by Eq.(3), the maximum correlation gives minimum value of present evaluation index.

$$\frac{1}{S}\int_S \left\{ \int_{s_k} \frac{Df}{Dt} ds \right\}^2 dS = \frac{1}{S}\int_S (f_{k+1} - f_k)^2 dS = \frac{1}{S}\int_S (f_{k+1}^2 + f_k^2) dS - 2\int_S f_{k+1} \cdot f_k dS = 0 \tag{7}$$

2.2 Extension for the multi-time-step PIV

Multi-time-step PIV images have further information than the velocity field. Figure 3 shows the schematic view of multi-time-step images. The vectors illustrated in Fig.3 show the displacement vector at target point (x, y), and it will be changed when the flow field is unsteady. The pair images at $[t_k, t_k + 1]$ give a velocity vector by applying the conventional PIV analysis. The vectors shown here indicate the relationship between closest times, but it can be understood that they give the gradient at each point on a surface that covers all path lines in spatio-temporal space. Then the object of the multi-time-step PIV analysis is to obtain the surface where the constraint equation is satisfied everywhere on it.

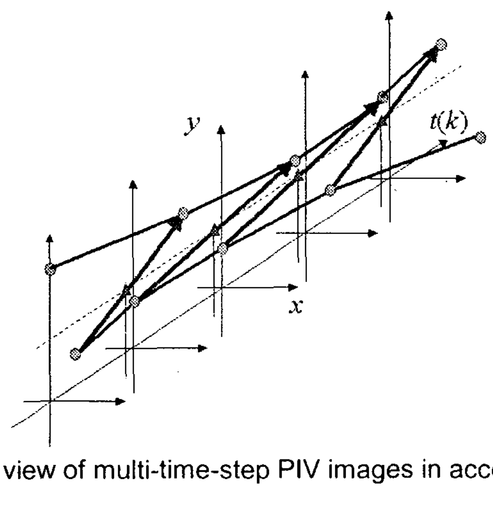

Fig.3 Schematic view of multi-time-step PIV images in accelerated flow field.

The minimum norm of the constraint equation gives optimum surface in spatio-temporal space, and it has been employed by gradient method. Equation (8) gives the analytical expression of evaluation index for the differential form of constraint equation. S and T show the area and period of measurement target, and the center of the area is fixed in space here.

$$\frac{1}{T\cdot S}\int_T\int_S\left(\frac{Df}{Dt}\right)^2 dSdt=0 \tag{8}$$

Conventional integral form of evaluation equation can be extended so as to apply it to the multi-time-step PIV images. When the path lines at target point (x,y) and time t are expressed by $s(t,x,y)$, Eq.(9) can be employed for the evaluation of multi-time-step images. When we can find the optimum set of path lines $s(t,x,y)$, they enable us to obtain the unsteady flow field.

$$\frac{1}{T\cdot S}\int_T\int_S\left\{\int_{s(t,x,y)}\frac{Df}{Dt}ds\right\}^2 dSdt=0 \tag{9}$$

The unknown parameters in Eqs.(8), (9) are the velocity field such as vectors or path lines and they are the function of (t,x,y). Usually the velocities in the correlation area are treated as uniform, but we can find another possibilities here to obtain further information.

3. Asymptotic approach for higher-order solution

The constraint equations and the evaluation indexes are described by means of differentiations and integrals of luminance function and they are defined at points or on line. In actual measurement, the differentiations and integrals are estimated by finite difference and numerical summation respectively. When we need to consider the higher order terms, all the equations cannot be solved at a time because the equations becomes non-liner. Asymptotic approaches enable us to get the higher-order information of flow field conventionally. The regular asymptotic algorithm has been applied on both the gradient method and the correlation method to achieve best performance with multi-time-step images.

3.1 Asymptotic approach for gradient method

<u>Finite difference form of constraint equation</u>

Present asymptotic approach assumes the form of solution with a regular series of functions. When the time interval of target image pair is represented with Δt, the displacements of particle image are expressed by Eqs.(10), (11) using the target point velocity (u_0, v_0) and its derivatives.

$$\Delta x=u_0\Delta t+\frac{1}{2}\left(\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+v_0\frac{\partial u}{\partial y}\right)(\Delta t)^2+O(\delta^3). \tag{10}$$

$$\Delta y=v_0\Delta t+\frac{1}{2}\left(\frac{\partial v}{\partial t}+u_0\frac{\partial v}{\partial x}+v_0\frac{\partial v}{\partial y}\right)(\Delta t)^2+O(\delta^3) \tag{11}$$

The differential form of constraint equation shown by Eq.(4) consists of differentiations of luminance function and velocity field information. The differential form equation can be obtained by considering the finite difference of luminance function by assuming the limitation $\Delta t\to 0$. Substituting Eqs.(10), (11) into the Taylor's expansion of luminance function, following basic relationship between the luminance function and velocity field can be obtained.

$$\begin{aligned}&f(t+\Delta t,x+\Delta x,y+\Delta y)-f(t,x,y)\\&=\left(\frac{\partial f}{\partial t}+u_0\frac{\partial f}{\partial x}+v_0\frac{\partial f}{\partial y}\right)\Delta t+\frac{1}{2}\frac{\partial f}{\partial x}\left(\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+v_0\frac{\partial u}{\partial y}\right)\Delta t^2+\frac{1}{2}\frac{\partial f}{\partial y}\left(\frac{\partial v}{\partial t}+u_0\frac{\partial v}{\partial x}+v_0\frac{\partial v}{\partial y}\right)\Delta t^2\\&+\frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}+u_0^2\frac{\partial^2 f}{\partial x^2}+v_0^2\frac{\partial^2 f}{\partial y^2}\right)\Delta t^2+\left(u_0\frac{\partial^2 f}{\partial t\partial x}+u_0v_0\frac{\partial^2 f}{\partial x\partial y}+v_0\frac{\partial^2 f}{\partial y\partial t}\right)\Delta t^2+O(\delta^3)\end{aligned} \tag{12}$$

Equation (12) contains following unknowns eight unknowns, $(u_0, \partial u/\partial t, \partial u/\partial x, \partial u/\partial y, v_0, \partial v/\partial t, \partial v/\partial x)$. The finite form constraint equation can be derived by substituting Eq.(12) into Eq.(4), which expresses the fact that the luminance at (t,x,y) and $(t+\Delta t, x+\Delta x, y+\Delta y)$ are same when external factors such as illumination and scattering characteristics change are negligible. As Eq.(13) is expressed until second-order terms, we can find several non-liner parts such as $u_0 \cdot \partial u/\partial x$, $v_0 \cdot \partial v/\partial x$ etc.

$$\begin{aligned}&\left(\frac{\partial f}{\partial t}+u_0\frac{\partial f}{\partial x}+v_0\frac{\partial f}{\partial y}\right)\Delta t+\frac{1}{2}\frac{\partial f}{\partial x}\left(\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+v_0\frac{\partial u}{\partial y}\right)\Delta t^2+\frac{1}{2}\frac{\partial f}{\partial y}\left(\frac{\partial v}{\partial t}+u_0\frac{\partial v}{\partial x}+v_0\frac{\partial v}{\partial y}\right)\Delta t^2\\&+\frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}+u_0^2\frac{\partial^2 f}{\partial x^2}+v_0^2\frac{\partial^2 f}{\partial y^2}\right)\Delta t^2+\left(u_0\frac{\partial^2 f}{\partial t\partial x}+u_0v_0\frac{\partial^2 f}{\partial x\partial y}+v_0\frac{\partial^2 f}{\partial y\partial t}\right)\Delta t^2=0\end{aligned} \tag{13}$$

Partial linear solution

The spatio-temporal method has used least-square approach for first-order solution of constraint equation of gradient method. When the time interval of pair image is sufficiently small, the terms higher than second-order can be negligible. Then the constraint equation contains only two unknowns (u_0, v_0), and the equation becomes linear. Then the least-square method can be directly applied, and the following accumulated equation can be obtained. Equation (14) can be applied on the region where the velocity field is uniform in time and space.

$$\begin{pmatrix}\sum\left(\frac{\partial f}{\partial x}\right)^2 & \sum\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\\ \sum\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & \sum\left(\frac{\partial f}{\partial y}\right)^2\end{pmatrix}\begin{pmatrix}u_0\\ v_0\end{pmatrix}=\begin{pmatrix}\sum\frac{\partial f}{\partial x}\frac{\partial f}{\partial t}\\ \sum\frac{\partial f}{\partial y}\frac{\partial f}{\partial t}\end{pmatrix} \tag{14}$$

The first step to the higher-order solution can be achieved by a linerizing approach similar to the first order solution. When the coupling between the velocity and its derivatives can be negligible, the constraint equation can be linearized as shown in Eq.(15).

$$\left(\frac{\partial f}{\partial t}+u_0\frac{\partial f}{\partial x}+v_0\frac{\partial f}{\partial y}\right)\Delta t+\frac{\partial f}{\partial x}\left(\frac{\partial u}{\partial t}\Delta t+\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y\right)\Delta t+\frac{\partial f}{\partial y}\left(\frac{\partial v}{\partial t}\Delta t+\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y\right)\Delta t=0$$

$$\frac{\partial f}{\partial t}+(u_0+\frac{\partial u}{\partial t}\Delta t+\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y)\cdot\frac{\partial f}{\partial x}+(\frac{\partial v}{\partial t}\Delta t+\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y)\cdot\frac{\partial f}{\partial y}=0 \tag{15}$$

As the coupling is ignored, the parameters Δt, Δx, Δy can be regarded as independent. Then the solution of Eq.(15) can be obtained by applying the least-square method. The cost function of the least-square method can be expressed by Eq.(16). Then the solution can be obtained by solving the linear equation array as same a Eq.(14).

$$E=\sum\left\{\frac{\partial f}{\partial t}+(u_0+\frac{\partial u}{\partial t}\Delta t+\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y)\cdot\frac{\partial f}{\partial x}+(\frac{\partial v}{\partial t}\Delta t+\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y)\cdot\frac{\partial f}{\partial y}\right\}^2 \tag{16}$$

Present approach actually improves accuracy of analysis by employing higher order fitting function for the solution plane (Fujita, 2003). Figure 4 shows the schematic view of the fitting function of least-square method. Usual first-order approach gives o-order solution, the uniform velocity field in spatio-temporal space, and present second-order approach enables us to obtain the first-order solution for the derivatives of velocities. However, the when we consider the application on the dynamic PIV system, original second-order constraint equation should be considered.

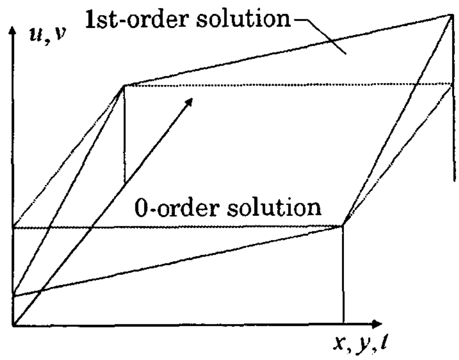

Fig.4 Solution planes of partial linear solution.

Asymptotic solution of non-liner constraint equation

The second-order constraint equation shown by Eq.(13) represent its non-liner characteristics as described before. The solutions of non-liner equation can be obtained by numerical approach such as Newton method, but the iteration approach is not practically applicable for actual image analysis, which requires heavy CPU time. Asymptotic approach to the non-liner equation enables us to obtain the solution with liner equation array.

Equation (13) shows the finite difference form of constraint equation. The first-order terms shows the same constraint equation of usual spatio-temporal derivative method, and the second-order terms shows the contribution of derivatives of velocity. When we can assume that the derivatives are smaller-order terms of the velocity, the constraint equation will balance in each orders as shown by Eqs.(17), (18).

$$\frac{\partial f}{\partial t}+u_0\frac{\partial f}{\partial x}+v_0\frac{\partial f}{\partial y}=0 \tag{17}$$

$$\begin{aligned}&\frac{\partial f}{\partial x}\left(\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}+v_0\frac{\partial u}{\partial y}\right)+\frac{\partial f}{\partial y}\left(\frac{\partial v}{\partial t}+u_0\frac{\partial v}{\partial x}+v_0\frac{\partial v}{\partial y}\right)\\&=\left(\frac{\partial^2 f}{\partial t^2}+u_0^2\frac{\partial^2 f}{\partial x^2}+v_0^2\frac{\partial^2 f}{\partial y^2}\right)+2\cdot\left(u_0\frac{\partial^2 f}{\partial t\partial x}+u_0v_0\frac{\partial^2 f}{\partial x\partial y}+v_0\frac{\partial^2 f}{\partial y\partial t}\right)\end{aligned} \tag{18}$$

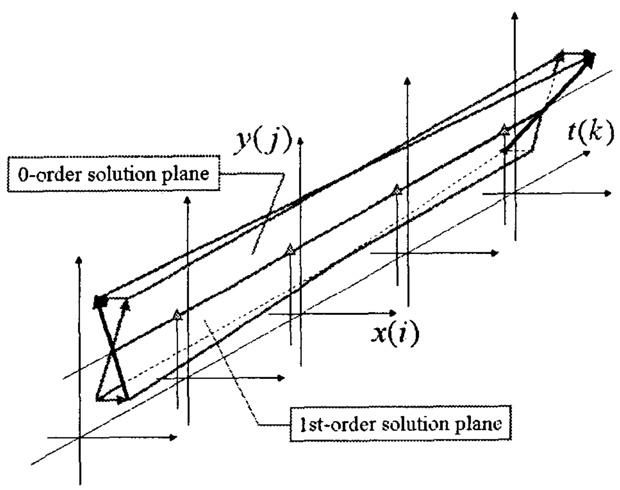

Fig.5 Schematic view for the asymptotic approach for the multi-time-step PIV.

Equation (17) shows exactly the same form of original differential style constraint equation shown when the uniform velocity field is assumed inside the correlation space. Equation (18) includes the first and second order of luminance derivatives, the averaged velocity (u_0, v_0) and the first order derivatives of velocity. When Eqs.(17),(18) are balanced independently as they are different order, initially we can determine (u_0, v_0) from Eq.(17). After we determine (u_0, v_0) with first-order equation, we apply it on Eq.(18). As (u_0, v_0) has already determine and the derivatives of velocity can be regarded as a constant value in the target region, Eq.(18) gives 1st-order derivatives of velocity field in time and space.

3.2. Asymptotic approach for cross-correlation method

The asymptotic approach is also available for the integral form of constraint equation. The correlation between the image at time $t = t_k$ and $t = t_{k+l}$ can be express by Eq.(19), that shows an evaluation index on a volume in spatio-temporal space. The evaluation index C_{kl} shows the similarity of pair image between two particular times, but multi-time-step PIV requires to evaluate total information in time and space.

$$C_{kl}(\Delta x, \Delta y) = \int_S f_k(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}) \cdot f_{k+l}(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}) dS \qquad (19)$$

The total evaluation index for multi-time-step images can be given by Eq.(20). Every combination of images in multi-time-step is covered with the index. The most primitive way of present approach is to select $M = 1$ that means we consider the relationship between the closest times. It seems to have no difference from the accumulation of conventional PIV technique, but it is different as we evaluate all the image data together. Figure 6 shows an example of time-sequence of correlation map. Each correlation map was obtained by analyzing the VSJ standard image. The minimum evaluation index gives optimum path line shown by Eq.(21). Arbitrary path lines is able to be the candidates for the solution, but it is not easy to determine without parametric consideration.

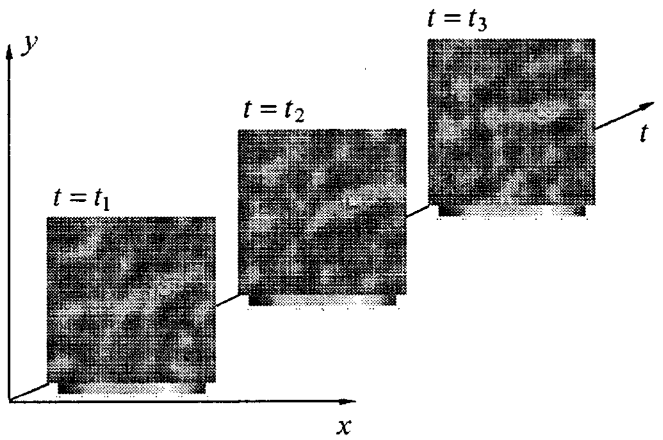

Fig.6 Correlation maps obtained by analyzing VSJ standard image.

$$\overline{C}(\Delta x, \Delta y) = \frac{1}{M \cdot N} \sum_{k=1}^{N} \sum_{l=1}^{M} C_{kl}(\Delta x, \Delta y) \qquad (20)$$

$$\max_{g \in L(\Delta x, \Delta y)} \left\| \overline{C}(\Delta x, \Delta y) \right\| = \left\| \overline{C}(\Delta x_{LS}, \Delta y_{LS}) \right\| \qquad (21)$$

The path line can be expressed by a Taylor's series with Δt. When we the displacement is written by vector form $\vec{s}(\Delta t) = (\Delta x, \Delta y)$, Eq.(22) shows Taylor's series of path lines where $\vec{v}^{(0)}$, $\vec{v}^{(1)}$, $\vec{v}^{(2)}$ show the averaged velocity vector and its temporal derivatives.

$$\vec{s}(\Delta t) = \vec{v}^{(0)} \Delta t + \frac{\vec{v}^{(1)}}{2!} \Delta t^2 + \frac{\vec{v}^{(2)}}{3!} \Delta t^3 + O(\delta^4) \qquad (22)$$

Substituting Eq.(22) into Eq.(19), the evaluation index expressed by Eq.(23) is able to be obtained. The evaluation equation includes finite number of parameter, and it enables us to obtain the optimum path line by determining the parameters one by one. The case of $K=0$ shows the conventional correlation method, and it gives the averaged velocity in correlation volume in spatio-temporal space. Considering the asymptotic approach as the same in the gradient method, we can assume that the lower order solution can be used to analyze the higher order. Initially we can find optimum $\vec{v}^{(0)}$ by considering whole the time-step images without consideration of higher-order terms. After we determine $\vec{v}^{(0)}$, we are able to find the optimum combination of $(\vec{v}^{(0)}, \vec{v}^{(1)})$ by considering Eq.(20) again. The most primitive way is to select $M=1$ in Eq.(19) that means we consider the relationship of pair images side by side, but there remains much possibility to consider the images with larger number of M, and non-uniform time interval.

$$\max_{g \in L(\vec{v}_K)} \left\| \overline{C}\left(\sum_{m=0}^{K} \vec{v}_m \frac{\Delta t^{m+1}}{(m+1)!} \right) \right\| = \left\| \overline{C}(\vec{s}_{LS}^{(K)}) \right\| \tag{23}$$

The spatial derivatives of velocity field does not appear positively in Eq.(21), but it could be included in the variance of velocity vectors $\vec{v}_k(x,y)$. The variance of velocity vector in space is also able to be expressed by Taylor series with $(\Delta x, \Delta y)$, and it can be evaluated in the integral through a correlation plane as same in Eq.(21).

6. Conclusion

Asymptotic approach for the analysis of multi-time-step PIV was proposed. The analytical basis of measurement shows how we can extend the PIV algorithms so as to apply multi-time-step images. Both the differential and integral approaches are considered to obtain the higher order solutions. Present approach enables us to obtain further information of PIV image appropriately. The advantage of present approach will contribute not only to the robustness, but also to keep consistency on all the results obtained from the multi-time-step images.

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