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Natural Frequencies and Mode Shapes of Beams with Step Change in Cross-Section

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ABSTRACT: *Natural frequencies of the transverse vibration of beams with step change in cross-section are obtained by using the asymptotic closed form solution. This closed form solution is found by using WKB method under the assumption of slowly varying properties, such as mass, cross-section, tension etc., along the beam length. However, this solution is found to be still very accurate even in the case of large variation in cross-section and tension. Therefore, this result can be easily applied to many engineering problems.*

1 Introduction

Modern Industry extensively use beams with variable properties in many structures and machineries for different purposes. The analytic solutions for the transverse vibration of the Euler-Bernoulli Beam with constant properties are well known in the literature. However, the closed form solutions for the variable properties along it's length, such as variable cross-section, tension, mass etc., have not been available until the author [1] found asymptotic closed form solutions by WKB method and published in the papers [2, 3]. The validity of these solutions is based on the assumption of slowly varying properties along the beam length. However, these solutions were found to be very accurate though the variation of tension was large [2]. In the limit when the variation goes to zero, these asymptotic solutions become exact solutions.

In many papers [4, 5] to find natural frequencies of beams with step changes in cross-section subject to different boundary conditions, the analytic solution for the constant cross-section in each segment and compatibility conditions at the junction points are used. Brief reviews on the vibration of beams with step changes in cross-section are as follows:

The frequency equation of a simply supported stepped beam was deduced in [8] by Levinson. Jang and Bert [9, 10] derived the frequency equations for all combinations of boundary conditions in the form of fourth order determinant equated to zero. The finite element method and commercial code were used to obtain the natural frequencies of a beam with circular cross-section. Naguleswaran presented a scheme to derive frequency equation and obtain natural frequencies and mode shape for all combinations of the classical boundary conditions by using bisection method. The first three frequencies and sensitivity of the frequency parameters were tabulated for several combinations of system parameters. The results were extended for the beams with up to three step changes in cross-section [5].

In the present paper the first three natural frequencies of circular bipinned beam with one step change in cross-section are tabulated for the different beam parameters. The frequencies are obtained by using the asymptotic closed form solutions for the large variation case, such as step change in cross-section. As a result these solutions are still found to be very accurate even for the case of step change in cross-section and can be easily applied to many beam vibration problems.

2 Asymptotic solution by WKB method

The derivation of the asymptotic solutions by using WKB method is described in details in [1] and published in [2]. The governing equation and the asymptotic solution for the transverse vibration of the beam with variable properties are as follows:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 W}{\partial x^2}\right]-\frac{\partial}{\partial x}\left[T(x)\frac{\partial W}{\partial x}\right]+m(x)\frac{\partial^2 W}{\partial t^2}=0. \quad (1)$$

By introducing the non-dimensional quantities which are defined in Appendix A, the equation becomes

$$\frac{\partial^2}{\partial s^2}\left[P(s)\frac{\partial^2 Y}{\partial s^2}\right]-\frac{\partial}{\partial s}\left[Q(s)\frac{\partial Y}{\partial s}\right]+U(s)\frac{\partial^2 Y}{\partial \tau^2}=0. \quad (2)$$

After the separation of variables $Y(s,\tau)=R(s)H(\tau)$ the equation (2) can be reduced to

$$\frac{d^2}{ds^2}\left[P(s)\frac{d^2 R}{ds^2}\right]-\frac{d}{ds}\left[Q(s)\frac{dR}{ds}\right]-U(s)\Lambda^2 R = 0 \quad (3)$$

$$\ddot{H}+\Lambda^2 H = 0 \quad (4)$$

An asymptotic closed-form solution of the above equation was obtained as follows

$$R(s)=T_1(s)\left[C_1\sin\left\{\int_0^S h_2 d\xi\right\}+C_2\cos\left\{\int_0^S h_2 d\xi\right\}\right]$$
$$+T_2(s)\left[C_3\sinh\left\{\int_0^S h_1 d\xi\right\}+C_4\cosh\left\{\int_0^S h_1 d\xi\right\}\right], \quad (5)$$

where C_1, C_2, C_3, C_4 are constants and $R(s)$, s are non-dimensional displacement and axial coordinate, respectively, and $T_1(s), T_2(s), h_1(s), h_2(s)$ are assumed to be slowly varying quantities and defined in the Appendix A. In order to find natural frequencies of a beams, the following boundary conditions can be considered.

3 Frequency equations and Mode Shapes

3.1 Simply Supported Beams

The simply supported boundary conditions are

$$R(0)=R''(0)=R(1)=R''(1)=0, \quad (6)$$

where a prime denotes a derivative with respect to s.

By substituting (6) into (5), the following simple, asymptotic formulas to predict natural frequencies and mode shapes of the beam can be obtained:

$$\int_0^1 \sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda_n^2}{P}}}\,d\xi=n\pi, \quad n=1,2,\dots \quad (7)$$

In dimensional form,

$$\int_0^L \sqrt{-\frac{1}{2}\left(\frac{T(x)}{EI(x)}\right)+\frac{1}{2}\sqrt{\left(\frac{T(x)}{EI(x)}\right)^2+4\frac{m(x)\omega_n^2}{EI(x)}}}\,dx = n\pi, \quad n=1,2,\dots \quad (8)$$

Mode shapes are

$$R_n(s)=\sin\left\{\int_0^S \sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda_n^2}{P}}}\,ds\right\}. \quad (9)$$

Furthermore, the orthonormal characteristic functions becomes

$$\phi_n(S)=\frac{\sin\left\{\int_0^S\sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda_n^2}{P}}}\,ds\right\}}{\int_0^1 U(s)\sin^2\left\{\int_0^S\sqrt{-\frac{1}{2}\left(\frac{Q}{P}\right)+\frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2+4\frac{U\Lambda_n^2}{P}}}\,ds\right\}ds}. \quad (10)$$

3.2 Fixed-Fixed beams

The boundary conditions are

$$R(0)=R'(0)=R(1)=R'(1)=0. \quad (11)$$

The natural frequencies and mode shapes in this case can be obtained by solving

$$\det M_{fx-fx}=0 \quad (12)$$

where matrix M is defined in the Appendix A.

3.3 Free-Free Beams

The corresponding boundary conditions are given by

$$R''(0)=R'''(0)=R''(1)=R'''(1)=0. \quad (13)$$

Similarly, the characteristic equation becomes

$$\det M_{fr-fr}=0 \quad (14)$$

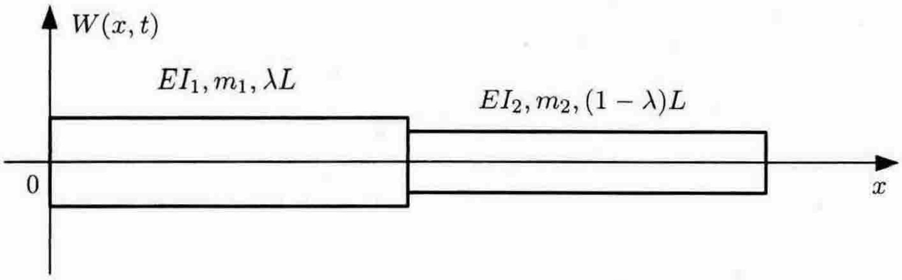

Figure 1: The beam with one-step change in cross-section

3.4 Sliding-Sliding Beams

The boundary conditions are

$$R'(0) = R'''(0) = R'(1) = R'''(1) = 0. \quad (15)$$

The characteristic equation becomes

$$\det M_{sl-sl} = 0 \quad (16)$$

4 Natural Frequencies of stepped beam

The Euler-Bernoulli beam with one step change in cross-section at $x=\lambda L$ is considered as shown in Fig. 1. The step location divides the beam into two sections with flexural rigidities EI_1, EI_2, masses per unit length m_1, m_2 and lengths $\lambda L, (1-\lambda)L$, respectively.

To find analytical expression for natural frequencies the beam parameters are assumed to be slow varying along the length. In case of step change in cross-section the $I(x)$ and $m(x)$ becomes step functions

$$I(x) = \begin{cases} I_1, & \text{if } 0 \le x \le \lambda L, \\ I_2, & \text{if } \lambda L < x \le L, \end{cases} \quad (17)$$

$$m(x) = \begin{cases} m_1, & \text{if } 0 \le x \le \lambda L, \\ m_2, & \text{if } \lambda L < x \le L. \end{cases} \quad (18)$$

The equation of motion is

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) + m(x)\frac{\partial^2 y}{\partial t^2} = 0. \quad (19)$$

After introducing non-dimensional quantities as in Appendix, A the governing equation becomes

$$\frac{\partial^2}{\partial s^2}\left(P(s)\frac{\partial^2 Y}{\partial s^2}\right) + U(s)\frac{\partial^2 Y}{\partial t^2} = 0. \quad (20)$$

The non-dimensionalized frequencies α_1, α_2, flexural rigidity ratio ι and mass per unit length ratio μ are defined as follows

$$\alpha_1 = \sqrt[4]{\frac{\omega^2 m_1 L^4}{EI_1}}, \quad \alpha_2 = \sqrt[4]{\frac{\omega^2 m_2 L^4}{EI_2}} = \left(\frac{\mu}{\iota}\right)^{1/4}\alpha_1,$$
$$\iota = \frac{EI_2}{EI_1}, \quad \mu = \frac{m_2}{m_1}, \quad (21)$$

After the separation of variables $Y(s,t)=R(s)H(\tau)$, asymptotic solution for $R(s)$ is

$$R(s) = T(s)\left[C_1 \sin\left(\int_0^s h(\xi)d\xi\right) + C_2 \cos\left(\int_0^s h(\xi)d\xi\right) + C_3 \sinh\left(\int_0^s h(\xi)d\xi\right) + C_4 \cosh\left(\int_0^s h(\xi)d\xi\right)\right], \quad (22)$$

where

$$T(s) = \frac{1}{\sqrt{P(s)}}\left\{\frac{1}{2}\left[4\frac{U(s)\omega}{P(s)}\right]^{\frac{3}{2}}\right\}^{-\frac{1}{4}} \quad (23)$$

Table 1: D function for the various combination of boundary conditions

D	clamped (cl)	pinned (pn)	sliding (sl)	free (fr)
cl	$2-2B_4B_2$	$2B_1B_4-2B_2B_3$	$2B_2B_3+2B_1B_4$	$2+2B_2B_4$
pn	$2B_2B_3-2B_1B_4$	$-4B_1B_3$	$-4B_2B_4$	$2B_1B_4-2B_2B_3$
sl	$-2B_2B_3-2B_1B_4$	$-4B_2B_4$	$4B_1B_3$	$2B_2B_3+2B_1B_4$
fr	$2+2B_2B_4$	$2B_2B_3-2B_1B_4$	$-2B_2B_3-2B_1B_4$	$2-2B_4B_2$

$$h(s) = \left(\frac{U(s)\omega}{P(s)}\right)^{\frac{1}{4}}. \quad (24)$$

The detailed derivation can be found in [1]. To obtain frequency equation, the boundary conditions must be imposed, then general expression of frequency equation becomes in the following form.

$$T^2(0)T^2(1)h^m(0)h^n(1)D(B_1, B_2, B_3, B_4) = 0, \quad (25)$$

where D is a function of B_1, B_2, B_3, B_4. The form of D depends on the combination of boundary conditions. m and n are overall ranks of boundary conditions. All possible combinations of boundary conditions have been solved. The result is shown in Table 1. In the case of simply supported beam the equation (25) becomes

$$T^2(0)T^2(1)h^2(0)h^2(1)\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ B_1 & B_2 & B_3 & B_4 \\ -B_1 & -B_2 & B_3 & B_4 \end{vmatrix}$$
$$= -T^2(0)T^2(1)h^2(0)h^2(1)4B_1B_3 = 0. \quad (26)$$

In general, terms $T^2(0), T^2(1), h^2(0), h^2(1)$ are not equal to zero. Hence the equation above is reduced to

$$4\sin\left(\int_0^1 h(\xi)d\xi\right)\sinh\left(\int_0^1 h(\xi)d\xi\right) = 0, \quad (27)$$

or in dimensional form

$$\int_0^L \left\{\frac{m(x)\omega_n^2}{EI(x)}\right\}^{\frac{1}{4}} dx = n\pi, \quad n = 1, 2, \ldots. \quad (28)$$

After solving the equation the analytical expression for natural frequencies becomes

$$\omega_n = \left(\frac{n\pi}{L}\right)^2 \frac{1}{\left(\lambda\left[\frac{m_1}{EI_1}\right]^{1/4} + (1-\lambda)\left[\frac{m_2}{EI_2}\right]^{1/4}\right)^2}. \quad (29)$$

The integration has been done by using the fact of jump in values of functions $m(x), I(x)$ at step location. The analytical expression for non-dimensional frequency of first section is shown as follows

$$\alpha_{1_n} = \frac{n\pi}{\lambda + \left(\frac{\mu}{\iota}\right)^{1/4}(1-\lambda)}, \quad n = 1, 2, \ldots. \quad (30)$$

The ratio $\frac{\mu}{\iota}$ depends on type of the cross-section. In the case of circular cross-section $\mu = d^2$ and $iota = d^4$, where d denotes the ratio between dimensions of the first and second sections. Hence, equation (30) becomes

$$\alpha_{1_n} = n\pi\frac{\sqrt{d}}{(\sqrt{d}-1)\lambda + 1}, \quad n = 1, 2, \ldots. \quad (31)$$

Note that d=1 or λ=1 means no step change in cross-section. In such cases (31) becomes $n\pi$, exact natural frequencies for ordinary Euler-Bernoulli bipinned beam without step change in cross section.

The frequency α_{1_n} is the function of d, λ, the parameters of step change. The dependency factor α

$$\alpha(d, \lambda) = \frac{\sqrt{d}}{(\sqrt{d}-1)\lambda + 1}$$

carries the distortion due to step change. α is increasing function with respect to d and λ. α is convex function with respect to d and concave with respect to λ.

$$\frac{\partial\alpha}{\partial d} > 0, \quad \frac{\partial\alpha}{\partial\lambda} > 0, \quad \frac{\partial^2\alpha}{\partial d^2} < 0, \quad \frac{\partial^2\alpha}{\partial\lambda^2} > 0.$$

with property

$$\alpha(1, \lambda) = \alpha(d, 1) = 1.$$

To estimate the accuracy of the asymptotic solution, the results are compared with the numerical scheme suggested in [4] For the first three natural frequencies α_{1_n} the error is less 5% when d and λ are close to 1. Error has tendency to decrease when λ goes to one or zero.

The first three natural frequencies of the circular bipinned beam are tabulated in Tab. 2, 3, 4.

The frequencies for another types of cross section can be found by using the same procedure. However, only four combinations of the sliding and pinned boundary conditions allows to obtain analytical expression for natural frequencies in a form of

$$\alpha_n = f(n)\frac{1}{\lambda + \left(\frac{\mu}{\iota}\right)^{1/4}(1-\lambda)}, \quad n = 1, 2, \ldots,$$

where $\frac{\mu}{\iota}$ can be expressed in terms of d and depends on type of cross-section.

5 Conclusions

The first three natural frequencies of circular bipinned beam with one step change in cross-section are tabulated for the different beam parameters. The frequency equations for all combinations of the boundary conditions were derived and analytical expressions for frequencies for the several boundary conditions have been obtained. The frequencies were compared with exact values which were calculated numerically. The error is found to be very small for the step parameters closed to unity. As a result, the asymptotic solution is found to be very accurate even for the case of step change in cross-section and can easily applied to many beam vibration problem.

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A Appendix

The non-dimensional quantities are defined by

$$s = \frac{x}{\lambda}, \quad \tau = \omega_0 t, \quad \omega_0 = \sqrt{\frac{E_0 I_0}{M_0 \lambda^4}}, \quad Y = \frac{W}{D_e}, \quad L^* = \frac{L}{\lambda},$$

where subscript $_0$ denotes the reference section, λ is a characteristic length (e.g. the transverse wave length) and D_e is the effective diameter of the beam (or the width of the beam). For simplicity λ is chosen to be equal to L. The non-dimensional parameters are given by

$$P(s) = \frac{EI(Ls)}{E_0 I_0}, \quad Q(s) = \frac{T(Ls)L^2}{E_0 I_0},$$
$$U(s) = \frac{m(Ls)}{M_0}, \quad \Lambda_n = \frac{\omega_n}{\omega_0}.$$

$$T_1(S) = \frac{1}{\sqrt{P}}\left[\frac{1}{2}\left(\frac{Q}{P}\right) + 2\frac{QU\Lambda^2}{P^2} + \frac{1}{2}\left\{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda^2}{P}\right\}^{3/2}\right]^{-1/4}$$

$$T_2(S) = \frac{1}{\sqrt{P}}\left[-\frac{1}{2}\left(\frac{Q}{P}\right) - 2\frac{QU\Lambda^2}{P^2} + \frac{1}{2}\left\{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda^2}{P}\right\}^{3/2}\right]^{-1/4}$$

$$h_1(S) = \sqrt{\frac{1}{2}\left(\frac{Q}{P}\right) + \frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda^2}{P}}}$$

$$h_2(S) = \sqrt{\frac{1}{2}\left(-\frac{Q}{P}\right) - \frac{1}{2}\sqrt{\left(\frac{Q}{P}\right)^2 + 4\frac{U\Lambda^2}{P}}}$$

$$\begin{aligned} B_1 &= \sin \textstyle\int_0^1 h(\xi)d\xi \\ B_2 &= \cos \textstyle\int_0^1 h(\xi)d\xi \\ B_3 &= \sinh \textstyle\int_0^1 h(\xi)d\xi \\ B_4 &= \cosh \textstyle\int_0^1 h(\xi)d\xi \end{aligned}$$

$$M_{fx-fx} = \begin{vmatrix} 0 & T_2(0) & 0 & T_1(0) \\ T_2(0)h_2(0) & 0 & T_1(0)h_1(0) & 0 \\ T_2(1)B_1 & T_2(1)B_2 & T_1(1)B_3 & T_1(1)B_4 \\ T_2(1)B_2h_2(1) & -T_2(1)B_1h_2(1) & T_1(1)B_4h_1(1) & T_1(1)B_3h_1(1) \end{vmatrix}$$

$$M_{fr-fr} = \begin{vmatrix} 0 & -T_2(0)h_2^2(0) & 0 & T_1(0)h_1^2(0) \\ -T_2(0)h_2^3(0) & 0 & T_1(0)h_1^3(0) & 0 \\ -T_2(1)h_2^2(1)B_1 & -T_2(1)h_2^2(1)B_2 & T_1(1)h_1^2(1)B_3 & T_1(1)h_1^2(1)B_4 \\ -T_2(1)h_2^3(1)B_2 & T_2(1)h_2^3(1)B_1 & T_1(1)h_2^3(1)B_4 & T_1(1)h_2^3(1)B_3 \end{vmatrix}$$

$$M_{sl-sl} = \begin{vmatrix} T_2(0)h_2(0) & 0 & T_1(0)h_1(0) & 0 \\ -T_2(0)h_2^3(0) & 0 & T_1(0)h_1^3(0) & 0 \\ T_2(1)B_2h_2(1) & -T_2(1)B_1h_2(1) & T_1(1)B_4h_1(1) & T_1(1)B_3h_1(1) \\ -T_2(1)h_2^3(1)B_2 & T_2(1)h_2^3(1)B_1 & T_1(1)h_2^3(1)B_4 & T_1(1)h_2^3(1)B_3 \end{vmatrix}$$

parameters	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
$\lambda = 0.1$	2.48958 (2.43032)	2.67209 (2.62749)	2.83991 (2.81015)	2.99575 (2.98093)
$\lambda = 0.2$	2.54835 (2.41581)	2.71721 (2.62439)	2.87054 (2.81307)	3.01128 (2.98491)
$\lambda = 0.3$	2.60995 (2.40171)	2.76388 (2.628)	2.90183 (2.82505)	3.02698 (2.99575)
$\lambda = 0.4$	2.67461 (2.41113)	2.81218 (2.65253)	2.93382 (2.85321)	3.04284 (3.01548)
$\lambda = 0.5$	2.74256 (2.46385)	2.8622 (2.70984)	2.96652 (2.90175)	3.05886 (3.0436)
$\lambda = 0.6$	2.81404 (2.57431)	2.91403 (2.80532)	2.99995 (2.9688)	3.07506 (3.07642)
$\lambda = 0.7$	2.88936 (2.74929)	2.96777 (2.93169)	3.03415 (3.04321)	3.09143 (3.1074)
$\lambda = 0.8$	2.96881 (2.96226)	3.02354 (3.05632)	3.06914 (3.10486)	3.10797 (3.12964)
$\lambda = 0.9$	3.05276 (3.11245)	3.08143 (3.1287)	3.10494 (3.1363)	3.12469 (3.13994)

Table 2: First frequency, bracketed frequencies are obtained numerically

parameters	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
$\lambda = 0.1$	4.97917 (4.84919)	5.34418 (5.25288)	5.67982 (5.62302)	5.9915 (5.96528)
$\lambda = 0.2$	5.0967 (4.84702)	5.43442 (5.2813)	5.74107 (5.66245)	6.02256 (5.99541)
$\lambda = 0.3$	5.21991 (5.003)	5.52776 (5.42352)	5.80366 (5.77206)	6.05395 (6.05503)
$\lambda = 0.4$	5.34923 (5.33044)	5.62436 (5.66434)	5.86764 (5.91521)	6.08567 (6.11238)
$\lambda = 0.5$	5.48512 (5.70034)	5.7244 (5.86601)	5.93304 (5.99871)	6.11772 (6.13384)
$\lambda = 0.6$	5.62809 (5.79468)	5.82806 (5.87534)	5.99991 (5.99178)	6.15012 (6.13308)
$\lambda = 0.7$	5.77871 (5.64702)	5.93555 (5.81714)	6.06831 (5.99599)	6.18285 (6.1561)
$\lambda = 0.8$	5.93762 (5.67329)	6.04707 (5.92397)	6.13828 (6.10281)	6.21594 (6.21677)
$\lambda = 0.9$	6.10552 (6.09087)	6.16287 (6.19492)	6.20989 (6.24587)	6.24938 (6.27117)

Table 3: Second frequency, bracketed frequencies are obtained numerically

parameters	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
$\lambda = 0.1$	7.46875 (7.26884)	8.01627 (7.88511)	8.51972 (8.44478)	8.98725 (8.9564)
$\lambda = 0.2$	7.64505 (7.43005)	8.15163 (8.04598)	8.61161 (8.57842)	9.03385 (9.03442)
$\lambda = 0.3$	7.82986 (7.89624)	8.29164 (8.38315)	8.7055 (8.77205)	9.08093 (9.10767)
$\lambda = 0.4$	8.02384 (8.24199)	8.43654 (8.51639)	8.80146 (8.80478)	9.12851 (9.11326)
$\lambda = 0.5$	8.22767 (8.06323)	8.5866 (8.45112)	8.89955 (8.82986)	9.17659 (9.15867)
$\lambda = 0.6$	8.44213 (8.31736)	8.74209 (8.74393)	8.99986 (9.0394)	9.22517 (9.25151)
$\lambda = 0.7$	8.66807 (8.9247)	8.90332 (9.03501)	9.10246 (9.14458)	9.27428 (9.27505)
$\lambda = 0.8$	8.90643 (8.80347)	9.07061 (8.95981)	9.20742 (9.13535)	9.32391 (9.29677)
$\lambda = 0.9$	9.15828 (8.97144)	9.2443 (9.19497)	9.31483 (9.32173)	9.37407 (9.38999)

Table 4: Third frequency, bracketed frequencies are obtained numerically