Modeling of Time Delay Systems using Exponential Analysis Method

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Abstract: In this paper, very simple methods based on the exponential analysis are presented by which transfer function models for processes can easily be obtained. These methods employ step responses or impulse responses of the processes. These can also give a more precise transfer function model compared to the well-known graphical methods. Transfer functions are determined based on Prony method, which is one of the oldest and the most representative methods in the exponential analysis. Here, the method is reformed and applied to obtain the so-called low-order transfer function with pure time delay from the data of the step response. The effectiveness of the proposed method is examined through several numerical examples and experiments of the 2-tank level control process.

Keywords: transfer function, identification, step response, exponential analysis, exponential decay

1. INTRODUCTION

This paper is concerned with the problems associated with obtaining transfer function of the process from the experimental data of the step response or impulse response. In many control engineering text books which include the so-called classical control theory, control systems design procedures are stated by assuming the *a priori* knowledge concerning the transfer function of the controlled plant. However few books refer to the determination of the transfer function of the process from the step response of the system. The modeling of the process dynamics in the form of transfer function is frequently discussed in relation to the determination of PID tuning. For example, three-parameter transfer function models characterized by the static gain, the time constant and the dead time, or static gain and two different time constants are by far the most commonly used models in the papers of PID controller tuning and these three parameters can be determined graphically for monotone step response[1,2,3].

At present, most text books take the position that the determination of the transfer function should belong to the area of system identification and we have many effective techniques concerning process dynamics identification basically derived by the combination of ARMA model and the least squares concepts [4]. However it requires sufficient richness of the input and some numerical procedures to convert the obtained results to transfer function in the continuous form.

On the contrary, identification by step response can be executed by a simple setpoint change so that it does not require any special inputs for identification [5]. It means that such a method is able to apply to many practical cases if we have suitable means of identification. Unfortunately we have had no effective means to determine transfer function of the system directly from the data of the step response except the above-stated graphical methods which are very useful in some practical cases but not so effective in general higher order case because of the accuracy of the obtained model.

As is well known, many physical and biological phenomena can be described by first order linear differential equations whose solution has exponential decay [6,7]. Systems which have a single decay can be described by

$$y(t)=\alpha e^{-\lambda t}, t\ge 0 \tag{1.1}$$

This is the simplest case and we have a more general form with multiple decays

$$y(t)=\sum_{i=1}^{n}\alpha_i e^{-\lambda_i t}, t\ge 0 \tag{1.2}$$

The form (1.1) is called "monoexponential analysis form" and the form (1.2) is called "multiexponential analysis form". Obviously these forms correspond to the impulse response or the equivalent step response of SISO linear time invariant systems with distinct real eigenvalues. Hence, it is possible to determine the corresponding transfer functions if we can determine the parameters α_i and λ_i in eq.(1.2) from the measured output data Such an idea has been considered in the analysis of physical phenomena The method using this idea is called exponential analysis method[7].

In section 2, we will explain the Prony method which is known as one of the oldest but the most representative multiexponential analysis methods. Then we will show the relation between multiexponential analysis form and the expression by the transfer function. It is noted that the above stated method was published in 1795 and the concrete procedure of the method given in this report is actually based on the procedures described in [7, 8]. In section 3, we will discuss the procedure for determining the approximate transfer function model with pure time delay which is very much interested in process control area. In section 4, two numerical examples will be given to explain the proposed method. In section 5, some experimental results will be given by using 2-tank water-level control system.

2. PRONY'S METHOD

Let us consider a single input and single output linear time invariant plant with n distinct real characteristic roots. The step response of the plant is given as

$$\tilde{y}(t)=\sum_{i=1}^{n}\alpha_i e^{-\lambda_i t}+\gamma, t\ge 0 \tag{2.1}$$

where $-\lambda_i, i=1,\cdots,n,$ denote the n distinct negative real characteristic roots of the system and γ denotes the constant steady state of the plant. In the following, we assume that the value of γ is known. Then eq.(2.1) results in eq.(1.2) by assuming $y(t)=\tilde{y}(t)-\gamma$. Hence we treat eq.(1.2) hereafter.

Suppose that the output $y(t)$ be measured at $2n$ equidistant points: for example,

$y(0),\ y(T),\cdots,\ y\{(2n-1)T\}$

where T denotes the sampling period. Let

$$e^{-\lambda_i T}=x_i,\ y_j=y_j(jT)$$
$$i=1,\cdots,n,\ j=0,1,\cdots,2n-1 \tag{2.2}$$

Then we have the following $2n$ equations with respect to $2n$ unknown variables $\alpha_i, x_i, i=1,\cdots,n$.

$$\begin{aligned}
y_0&=\alpha_1+\cdots+\alpha_n\\
y_1&=\alpha_1 x_1+\cdots+\alpha_n x_n\\
&\vdots\\
y_k&=\alpha_1 x_1{}^k+\cdots+\alpha_n x_n{}^k\\
&\vdots\\
y_{2n-1}&=\alpha_1 x_1{}^{2n-1}+\cdots+\alpha_n x_n{}^{2n-1}
\end{aligned} \tag{2.3}$$

solved this problem dexterously as stated in the following. These equations are nonlinear equations and it is difficult to obtain the solution. Prony

Let $x_1,\cdots,x_n$ be the solutions of the following algebraic equation,

$$(x-x_1)(x-x_2)\cdots(x-x_n)=0 \tag{2.4}$$

or equivalent equation,

$$a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x_1+a_0=0,\ a_n=1 \tag{2.5}$$

If we can determine coefficients $a_0,\cdots,a_{n-1}$, we have

$$\lambda_i=-\frac{1}{T}\log x_i,\ i=1,\cdots,n \tag{2.6}$$

from eq.(2.2) and solutions x_i of eq.(2.5) .Thus the problem results in a need to determine the coefficients $a_0,\cdots,a_{n-1}$ in eq.(2.5). Prony solved this problem only by using the above stated $2n$ equations (2.3)[7]. Here we will treat this problem in a slightly different form compared to Prony's original method.

We multiply the k-th equation in eq.(2.3) by a_0,the $k+1$ th equation by a_1,…,the $k+n-1$ th equation by a_{n-1} and the $k+n$ th equation by $a_n=1$. If we then add up these n equations, we have

$$a_0 y_k+a_1 y_{k+1}+\cdots+a_n y_{k+n}=\sum_{i=1}^{n}\alpha_i x_i{}^k(a_0+a_1 x_i+\cdots+a_n x_i{}^n) \tag{2.7}$$

We have assumed that x_i are the solutions of eq.(2.5), Hence

$$a_0 y_k+a_1 y_{k+1}+\cdots+a_n y_{k+n}=0 \tag{2.8}$$

Starting successively with the $k+1,k+2,\cdots,k+n,\cdots$, we find that eqs.(2.3) and (2.5) imply the following linear equations:

$$a_0 y_j+a_1 y_{j+1}+\cdots+a_n y_{j+n}=0,\ j=k,k+1,\cdots,k+n,\cdots \tag{2.9}$$

Suppose that the number of equations (2.9) are greater than or equal to n. Then the following relation gives estimated values of parameters a_i:

$$\bar{a}=-(\bar{N}^T\bar{N})^{-1}\bar{N}^T\bar{y} \tag{2.10}$$

where

$$\bar{a}=\begin{bmatrix}a_0\\ \vdots\\ a_{n-1}\end{bmatrix},\bar{y}=\begin{bmatrix}y_{k+n}\\ \vdots\\ y_{k+2n}\\ \vdots\end{bmatrix},\bar{N}=\begin{bmatrix}y_k & y_{k+1} & \cdots & y_{k+n-1}\\ \vdots & \vdots & \cdots & \vdots\\ y_{k+n-1} & y_{k+n} & \cdots & y_{k+2n-1}\\ \vdots & \vdots & \cdots & \vdots\end{bmatrix} \tag{2.11}$$

Here we assume the regularity of $\bar{N}^T\bar{N}$. From this result, we can determine n distinct roots x_i of eq.(2.5). It follows that the parameters α_i in eq.(2.3) can easily be determined. For example, here we will give a similar algorithm as shown in eq.(2.10). That is,

$$\bar{\alpha}=(\bar{X}^T\bar{X})^{-1}\bar{X}^T\bar{y} \tag{2.12}$$

where

$$\bar{\alpha}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix},\bar{y}=\begin{bmatrix}y_k\\ \vdots\\ y_{k+n}\\ \vdots\end{bmatrix},\bar{X}=\begin{bmatrix}x_1^k & x_2^k & \cdots & x_n^k\\ \vdots & \vdots & \cdots & \vdots\\ x_1^{k+n} & x_2^{k+n} & \cdots & x_n^{k+n}\\ \vdots & \vdots & \cdots & \vdots\end{bmatrix} \tag{2.13}$$

Here we assume the nonsingularity of the matrix $\bar{X}^T\bar{X}$. Prony's original method used first n equations in eq,(2.3) to determine $\bar{a}$ and the latter half n equations were used to determine $\bar{\alpha}$. However it is suggested in the literature [7] that such a selection is not so effective under the existence of noise. Hence the least squares approach is derived here as given in eqs.(2.10) and (2.12).

Transfer function of the plant can easily be determined from the obtained data. Let eq.(2.1) be the output of the unit step input of the corresponding plant. Then the Laplace Transform of eq.(2.1) becomes

$$\tilde{y}(s)=\sum_{i=1}^{n}\frac{\alpha_i}{s+\lambda_i}+\frac{\gamma}{s} \tag{2.14}$$

In eq.(2.14), we have assumed that the initial value of the plant is to be zero. It must include

$$\sum_{i=1}^{n}\alpha_i+\gamma=0 \tag{2.15}$$

Then we can calculate the transfer function as follows:

$$G(s)=\frac{\tilde{y}(s)}{u(s)} \tag{2.16}$$

Case of $n=2$ is shown as an example of eq.(2.16):

$$G(s)=\frac{-(\alpha_1\lambda_1+\alpha_2\lambda_2)s+\gamma\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)} \tag{2.17}$$

3. APPROXIMATION BY FIRST-ORDER SYSTEM+TIME DELAY

In process systems, the model

$$G(s)=\frac{K}{1+\bar{T}s}e^{-Ls} \tag{3.1}$$

is known as one of the most common model used in PID controller tuning. This system is characterized by three parameters: the static gain K, the time constant $\bar{T}$, and the dead time L. Here we discuss this model in detail because it has been
widely used practically.

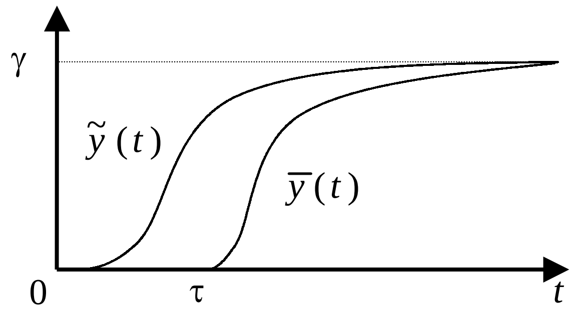

Fig.1 Approximation by low-order system

Consider the following model:

$$\begin{aligned}&\hat{y}(t)=\alpha e^{-\lambda(t-\tau)}+\gamma, t\geq\tau\\ &y(t)=\hat{y}(t)-\gamma=\bar{\alpha}e^{-\lambda t}, t\geq\tau\\ &\bar{\alpha}=\alpha e^{\lambda\tau}\end{aligned} \tag{3.2}$$

(1) Determination of $\lambda,\bar{\alpha}$

Let $e^{-\lambda T}=x$. Then

$$y_k=\bar{\alpha}x^k, k=k_0,k_0+1,\cdots,kT\geq k_0T\geq\tau \tag{3.3}$$

holds. Corresponding to eq.(2.5), we have

$$a_0+a_1x=0, a_1=1 \tag{3.4}$$

Multiplying a_0 to eq.(3.3) and using eq.(3.4) lead to

$$a_0y_k+y_{k+1}=0, k=k_0,k_0+1,\cdots,k_0+k_1, k_1\geq 1 \tag{3.5}$$

Solving the above k_1+1 relations yields the least squares solution of a_0. Parameter λ is then found as follows:

$$\lambda=-\frac{1}{T}\log(-a_0) \tag{3.6}$$

We also can determine the least squares value of parameter $\bar{\alpha}$ using k_1+1 relations of eq.(3.3).

(2) Determination of α,τ

From eq.(3.2), we have

$$\alpha=-\gamma, \tau=\frac{1}{\lambda}\log\frac{\bar{\alpha}}{\alpha} \tag{3.7}$$

Parameters (α,τ,λ) determines the three parameters of the model (3.1) such that

$$\bar{T}=\frac{1}{\lambda}, L=\tau, K=\gamma \tag{3.8}$$

It is noted that the values (α,τ,λ) are closely related to the selection of k_0: starting point of identification and k_1: section of matching. This raises a question concerning the best three parameters approximate model in eq.(3.1). This problem will be considered a little bit later through the experimental results.

4. EXAMINATION BY SECOND- ORDER SYSTEMS

In this section, several simple numerical examples are shown by using second order systems.

Example 1. Real distinct characteristic roots.

The model is

$$G(s)=\frac{1}{(6.7s+1)(2.4s+1)} \tag{4.1}$$

The exponential analysis was applied to the unit step response of eq.(4.1).Data was obtained from the section $t\in[0,5]$. Sampling interval is $T=0.1$ so that 500 points were used. Parameters $a_0,a_1,(a_2=1)$ were obtained from eq.(2.10) by substituting

$$\bar{a}=\begin{bmatrix}a_0\\a_1\end{bmatrix}, \bar{y}=\begin{bmatrix}y_2\\y_3\\\vdots\\y_{500}\end{bmatrix}, \bar{N}=\begin{bmatrix}y_0 & y_1\\y_1 & y_2\\\vdots & \vdots\\y_{498} & y_{499}\end{bmatrix} \tag{4.2}$$

α_1,α_2 can be determined by using eq.(2.11) where

$$\bar{\alpha}=\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}, \bar{y}=\begin{bmatrix}y_0\\y_1\\\vdots\\y_{500}\end{bmatrix}, \bar{X}=\begin{bmatrix}1 & 1\\x_1 & x_2\\\vdots & \vdots\\x_1^{500} & x_2^{500}\end{bmatrix} \tag{4.3}$$

Calculating results are as follows:

$$\begin{aligned}&a_0=0.945, a_1=-1.94, x_1=0.986, x_2=0.958\\ &\lambda_1=0.141, \lambda_2=0.424, \alpha_1=-1.49, \alpha_2=0.494\\ &\gamma=1\end{aligned} \tag{4.4}$$

Hence

$$\tilde{G}(s)=\frac{1.85\times10^{-4}s+5.96\times10^{-2}}{s^2+0.565s+5.96\times10^{-2}} \tag{4.5}$$

On the other hand, from eq.(4.1),

$$G(s)=\frac{6.19\times10^{-2}}{s^2+0.565s+6.19\times10^{-2}} \tag{4.6}$$

In Fig.2, step responses are derived for $G(s)$ and $\tilde{G}(s)$. The small term $1.85\times10^{-4}s$ in the numerator of $\tilde{G}(s)$ is thought to be a kind of numerical round off error which happened during the process of identification. The effects of such kinds of erroneous term will be discussed later.

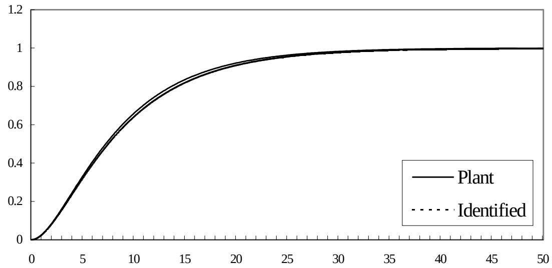

Fig.2 Step response of the plant with distinct roots

Example 2. Conjugate complex roots.

In the application of exponential analysis method, we have assumed that the characteristic roots are real and distinct. However, conjugate complex cases do not conflict with the distinct root assumption as long as we execute the calculation on the complex domain. Here we apply the exponential method to the second order vibrating system given by

$$G(s)=\frac{1}{s^2+2.51s+39.5} \tag{4.7}$$

Data is taken from the section $t\in[0.25,10]$ with sampling interval: $T=0.01$. Results are as follows.

$$\begin{aligned} &a_0=0.975, a_1=-1.97,\\ &x_1=0.986+j6.09\times10^{-2}, x_2=0.986-j6.09\times10^{-2}\\ &\alpha_1=-1.26\times10^{-2}+j2.61\times10^{-3}, \alpha_2=-1.26\times10^{-2}-j2.61\times10^{-3}\\ &\lambda_1=1.26+j6.34, \lambda_2=1.26-j6.34\\ &\gamma=1 \end{aligned} \tag{4.8}$$

$$\tilde{G}(s)=\frac{6.39\times10^{-2}s+1}{s^2+2.51s+39.6} \tag{4.9}$$

The step responses of $G(s)$ and $\tilde{G}(s)$ are given in Fig.3. There exists a small phase difference between ideal response and estimated response. We rewrite the transfer function as follows:

$$\tilde{G}(s)=\frac{1}{s^2+2.51s+39.6}+\frac{6.39\times10^{-2}s}{s^2+2.51s+39.6}=\tilde{G}_0(s)+\Delta\tilde{G}(s) \tag{4.10}$$

where $\Delta\tilde{G}(s)$ can be recognized as an additive mismatch. Ignoring $\Delta\tilde{G}(s)$ gives good matching of the response. It is noted that can see the same tendency in Example 1(Fig.4)

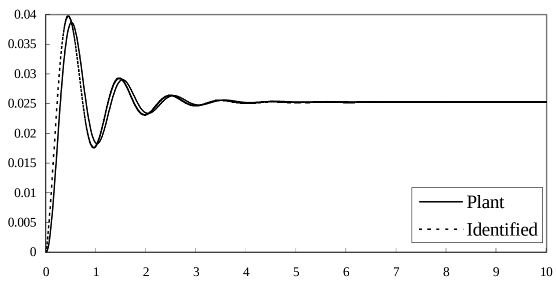

Fig.3 Simulation of the plant with conjugate complex

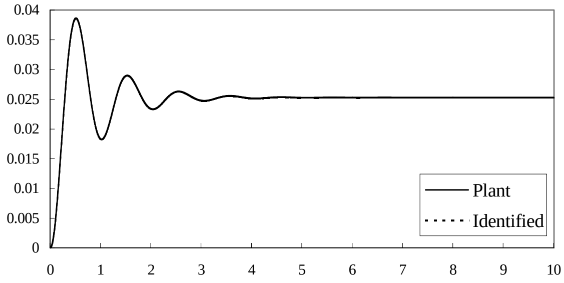

Fig.4 Simulation of the plant with conjugate complex roots ($\Delta\tilde{G}(s)=0$)

5. EXPERIMENTAL EXAMINATION BY LIQUID-LEVEL SYSTEM

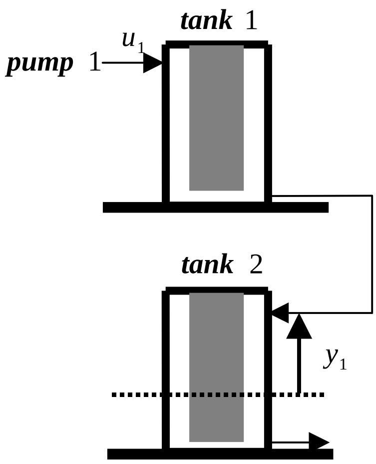

Fig.5 2-tank liquid level system

Consider the experimental system shown in Fig.5. The roblem is to obtain the transfer function between the water flow rate change $u(m^3/\text{sec})$ to tank 1 and the liquid level $y(t)$ of tank 2 from the steady state. In Fig.6, a step response used in the analysis is shown where $u=3.89\times10^{-6}(m^3/\text{sec})$.

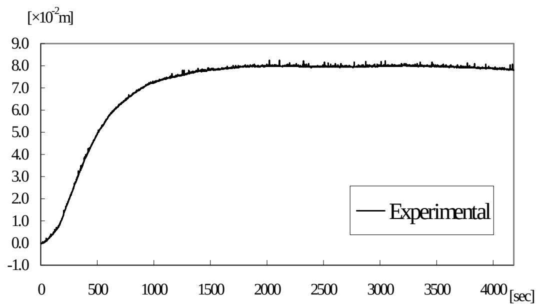

Fig.6 Step response of the liquid level of tank 2

5.1 Graphical approach as a second order system

To indicate and clarify the problems in classical techniques, a graphical method written in references [2,9] was applied to the step response given in Fig.6. This was to obtain a second order transfer function model. First we determine the inflection point A of the graph. Secondly, we write a tangent line at point A and seek for two intersecting points B and C with the final steady state level and the time axis. Then we can obtain two time constants of the second order model from the numerical values of B and C. An example of the model (5.1) and its step response (Fig.7) are shown below.

$$\tilde{G}(s) = \frac{2.01\times10^{4}}{2.7\times10^{4}s^{2}+510s+1} \tag{5.1}$$

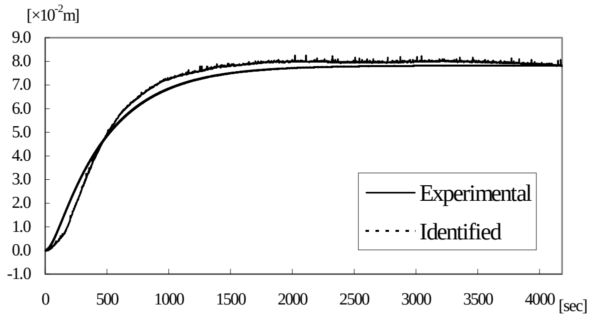

Fig.7 Step response of identified model by a graphical method

The difficulty of the method is that the result is quite sensitive to the selection of the inflection point and its tangent line. On the other hand, the exponential analysis gives a more compatible model compared to the model derived from the graphical data. It is noted that such a graphical method is not able to apply to higher order plants.

5.2 Approximation by first order system with dead time

This model is also known as the three parameter model [3]. The method given in section 3 is applied to the data in Fig. 6. Throughout the calculation, the sampling interval was fixed to 1 second. In the following, it is shown 4 different models corresponding to as 4 different matching sections.

(a) $t\in[100,2000]$

$$\tilde{G}(s) = \frac{44.1}{s+2.15\times10^{-3}}e^{-78s} \tag{5.2}$$

(b) $t\in[200,2000]$

$$\tilde{G}(s) = \frac{55.3}{s+2.70\times10^{-3}}e^{-136s} \tag{5.3}$$

(c) $t\in[350,2000]$

$$\tilde{G}(s) = \frac{61.7}{s+3.01\times10^{-3}}e^{-169s} \tag{5.4}$$

(d) $t\in[500,2000]$

$$\tilde{G}(s) = \frac{68.1}{s+3.32\times10^{-3}}e^{-212s} \tag{5.5}$$

Experimental data and simulation data of step responses for models (a),(b),(c) and (d) are shown in Fig.8.

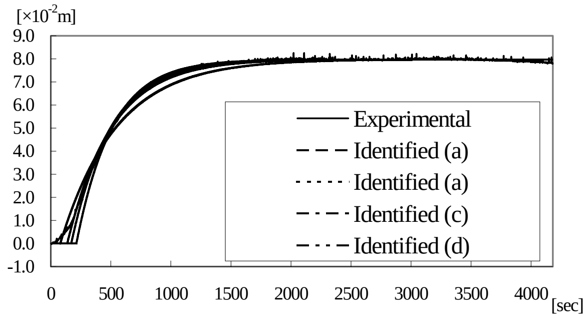

Fig.8 Step responses of approximate first order models with dead time

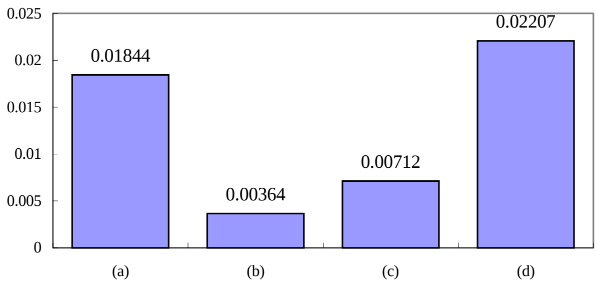

Fig.9 ISE between experimental curve and model

This is a quite natural question as to which model is the most relevant. Fig.11 expresses ISE between the experimental curve and the simulation curves. Here integration was executed on the section $t\in[0,2000]$. From the above mentioned facts, we can conclude that there exists some close relation between the compatibility of the model and the fitting sections of the curve. Actually, cases (b) and (c) realize well-fitted curves in Fig.10. However a method for realizing an "optimal" fitting is left for further consideration.

6. CONCLUSIONS

As one can see in this paper, very simple methods based on the exponential analysis were presented by which transfer function models for processes can easily be obtained. These can also give more precise transfer function model compared to well-known graphical methods in spite of using step response. Transfer functions are determined based on Prony method, which is one of the oldest and the most representative method in the exponential analysis. Here, the method is reformed and applied to obtain the so-called low-order with pure time delay transfer function from the data of the step response. The effectiveness of the proposed method is first examined through ideal second order numerical examples. Several basic possibilities concerning multiple root plant and vibratory system are given by simple numerical results. As an application to the practical system, modeling of the 2-tank liquid-level process from the experimental data of step response was considered and reasonable results were obtained in spite of the existence of measurement noise. The contents of this paper show only a few cases out of many possibilities, and as mentioned in the preceding sections, there are many interesting questions to be solved for practical applications.

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