Shape Design of Frame Structures for Vibration Suppression and Weight Reduction

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Abstract: This paper proposes shape design of frame structures for vibration suppression and weight reduction. The H_∞ norm of the transfer function from disturbance sources to the output points where vibration should be suppressed, is adopted as the performance index to represent the magnitude of vibration transfer. The design parameters are the node positions of the frame structure, on which constraints are imposed so that the structure achieves given tasks. For computation of Pareto optimal solutions to the two-objective design problem, a number of linear combinations of the H_∞ norm and the total weight of the structure are considered and minimized. For minimization of the scalared objective function, a Lagrange function is defined by the objective function and the imposed constraints on the design parameters. The solution for which the Lagrange function satisfies the Karush-Kuhn-Tucker condition, is searched by the sequential quadratic programming (SQP) method. Numerical examples are presented to demonstrate the effectiveness of the proposed design method.

Keywords: shape design, vibration suppression, weight reduction, Pareto optimal solution, H_∞ norm, model reduction

1. INTRODUCTION

In high-precision machines, vibration at the positioning parts has to be suppressed, which is caused by and transferred from other moving parts. On the other hand, reduction of the weight of a machine is commonly required for saving manufacturing and transportation costs. Since weight reduction generally contradicts vibration suppression, we need to deal with a two-objective design problem, which is practically very important.

In this context, we consider shape design of a frame structure. The design parameters are the node positions of the structure, on which constraints are imposed so that the structure achieves given tasks. We use the finite element method (FEM) to obtain the equation of motion of the structure.

To evaluate the vibration suppression property, we adopt the H_∞ norm of the transfer function from the disturbance sources to the displacement outputs, which we want to suppress. The H_∞ norm has been used as a performance index in structural design [1]∼[5]. Since the H_∞ norm is the maximum amplitude of the frequency response function, it can express the worst vibration transfer. This measure is suitable when the frequency range of the disturbance is broad. If the frequency of the disturbance is restricted in a specific band, a frequency-dependent weight should be introduced in the measure.

To compute Pareto optimal solutions to the two-objective design problem, we consider a number of linear combinations of the H_∞ norm and the total weight as the objective functions to be minimized. For minimization of the scalared objective function, we deal with a Lagrange function defined by the objective function and imposed constraints on the design parameters. We search the solution for which the Lagrange function satisfies the Karush-Kuhn-Tucker condition, by the sequential quadratic programming (SQP) method [6]. Since the derivative of the H_∞ norm with respect to the node positions of the frame structure cannot be calculated analytically, we compute the derivative numerically from the difference of H_∞ norms for small changes of node positions.

In the SQP method, we need to compute the H_∞ norm at each iteration. The computation time is long when the dimension of the model of the structure is high. For this reason, we first eliminate a large number of modes, which do not much contribute to the input-output relation, and then employ the balanced realization approach for model reduction [7].

Examples are presented to demonstrate the effectiveness of the proposed design method.

2. SYSTEM DESCRIPTION

We start modeling of a frame structure with the equation of motion obtained by FEM as

$$\left\{ \begin{array}{rcl} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) & = & Nw(t) \\ z(t) & = & Hq(t) \end{array} \right. \tag{1}$$

where q is the node displacement vector, w is the disturbance input vector, and z is the displacement output vector which we want to suppress. The matrices M, D, and K respectively denote mass, damping, and stiffness, which are positive definite real symmetric matrices. We derive the matrices M and K in Section 4.

We assume that the damping matrix D is proportional to the mass and stiffness matrices as

$$D = \alpha M + \beta K \tag{2}$$

where α and β are positive constants. This is a common assumption in mechanical structures [8]. In Eq.(1), the matrix N is defined by the locations and directions of the disturbance inputs, and H is defined by those of displacement outputs.

2.1. H_∞ Norm

To compute the H_∞ norm of the transfer function from the disturbance w to the output z, we transform Eq.(1) to the state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) \end{cases} \tag{3}$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ M^{-1}N \end{bmatrix}, \quad C = \begin{bmatrix} H & 0 \end{bmatrix}.$$

The transfer function, denoted by $G(s)$, is expressed as $C(sI-A)^{-1}B$ and its H_∞ norm is defined by

$$\|G(s)\|_\infty = \sup_\omega \overline{\sigma}\{G(j\omega)\} \tag{4}$$

where $\overline{\sigma}$ denotes the maximum singular value of the indicated matrix.

2.2. Model Reduction

Generally, state-space models Eq.(3) of structures are high-dimensional and it takes very much time to compute H_∞ norms of high-dimensional systems. Therefore, we consider model reduction.

The most popular technique of model reduction applicable to state equations is the one that uses the balanced realization [7]. However, it also takes much time to compute balanced realizations for high-dimensional state equations. For this reason, we propose elimination of insignificant modes in mode equations before we apply the balanced realization technique in the state space. Since mode equations are easily obtained from the equation of motion, this approach is efficient.

To do this, we diagonalize M and K in Eq.(1) so that

$$\begin{aligned} &T^T M T = I \\ &T^T K T = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\} \triangleq \Lambda \end{aligned} \tag{5}$$

where T is a nonsingular matrix and m is the dimension of the displacement vector q. Since M and K are positive definite real symmetric matrices, such a T always exists. Using this T, we transform Eq.(1) to

$$\begin{cases} \ddot{\hat{q}}(t) + \hat{D}\dot{\hat{q}}(t) + \Lambda\hat{q}(t) = \hat{N}w(t) \\ z(t) = \hat{H}\hat{q}(t) \end{cases} \tag{6}$$

$$\hat{q} = T^{-1}q, \quad \hat{D} = T^T D T, \quad \hat{N} = T^T N, \quad \hat{H} = HT.$$

From Eq.(6), the transfer function from w to z is written as

$$\begin{aligned} G(s) &= \hat{H}(s^2 I + s\hat{D} + \Lambda)^{-1}\hat{N} \\ &= \sum_{i=1}^{m} \hat{h}_i (s^2 + \hat{d}_i s + \lambda_i)^{-1}\hat{n}_i \end{aligned} \tag{7}$$

where $\hat{h}_i$, $\hat{n}_i$ denote the i-th column and row of $\hat{H}$, $\hat{N}$, respectively, and $\hat{D} = \mathrm{diag}\{\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_m\}$, $\hat{d}_i = \alpha + \beta\lambda_i$ $(i = 1, 2, \cdots, m)$. In this expression, the norm of the i-th component $G_i(s) = \hat{h}_i(s^2 + \hat{d}_i s + \lambda_i)^{-1}\hat{n}_i$ in the frequency domain is

$$\|G_i(j\omega)\| = \frac{\sqrt{\hat{h}_i^T\hat{h}_i}\sqrt{\hat{n}_i\hat{n}_i^T}}{\sqrt{(-\omega^2 + \lambda_i)^2 + \hat{d}_i^2\omega^2}}, \tag{8}$$

and its maximum with respect to ω is computed as

$$\|G_i(j\omega)\|_{\max} = \frac{2\sqrt{\hat{h}_i^T\hat{h}_i}\sqrt{\hat{n}_i\hat{n}_i^T}}{\hat{d}_i\sqrt{4\lambda_i - \hat{d}_i^{\,2}}}. \tag{9}$$

If this is large, the i-th mode reflects in the H_∞ norm of $G(s)$. If it is small, we delete the i-th mode from Eq.(6) to obtain a reduced order model. In this way, we can practically delete a large number of modes and obtain a significantly low order model.

We note however that even if $\|G_i(j\omega)\|_{\max}$ is not so large, if there is another mode nearby, then the total input-output properties of the two modes would reflect in the H_∞ norm of $G(s)$. Therefore, if there are modes, which are close and their maximum norms defined by Eq.(9) are not small, we do not delete them.

3. OPTIMAL DESIGN

Generally, a multi-objective optimization problem does not have an optimal solution. Instead, it has so-called Pareto optimal solutions, which have the property that it is impossible to reduce any objective function without increasing at least one of the other objective functions. In this paper, a Pareto optimal solution means that the H_∞ norm and the total weight determined by the solution are not simultaneously worse than those of any other solutions. They exist on the boundary of the region of feasible solutions and are not unique.

We formulate the two-objective structural optimization problem with the design parameters vector p composed of the changes of node positions from the nominal ones in the x, y and z directions. We impose upper and lower bounds on the changes of node positions so that the resultant structure does not violate the given task of the structure.

To compute a Pareto optimal solution, we consider a linear combination of the H_∞ norm $\|G(s,p)\|_\infty$ and the total weight $w(p)$ which are normalized by their nominal values. We write p explicitly to indicate that these quantities depend on the parameter. Thus, the problem is formulated as

$$\begin{aligned} &\min_p \quad f(p) = \gamma_1 \frac{\|G(s,p)\|_\infty}{\|G(s,0)\|_\infty} + \gamma_2 \frac{w(p)}{w(0)} \\ &\text{s.t.} \quad -p + \underline{p} \le 0, \quad p - \overline{p} \le 0 \end{aligned} \tag{10}$$

where γ_1, γ_2 are positive numbers such that $\gamma_1 + \gamma_2 = 1$, and $\underline{p}$, $\overline{p}$ are the lower and upper bound vectors for p. We obtain Pareto optimal solutions of the original two-objective problem by computing the optimal solutions of the scalared objective function Eq.(10) for various sets of the coefficients γ_1 and γ_2.

3.1. KKT Condition

To compute the optimal solution of Eq.(10), we consider a Lagrange function

$$L(p,\mu) = f(p) + \mu^T \begin{bmatrix} -p+\underline{p} \\ p-\overline{p} \end{bmatrix} \tag{11}$$

where μ is a Lagrange multiplier vector. Then, the optimization problem Eq.(10) is reduced to that of searching the parameter value p for which Eq.(11) satisfies the Karush-Kuhn-Tucker (KKT) condition [6].

$$\begin{aligned} &\nabla_p L(p,\mu) = \nabla f(p) + \mathrm{diag}\{-I,\, I\}\mu = 0 \\ &\nabla_\mu L(p,\mu) = \begin{bmatrix} -p+\underline{p} \\ p-\overline{p} \end{bmatrix} \le 0 \\ &\mu \ge 0 \\ &\mu^T \begin{bmatrix} -p+\underline{p} \\ p-\overline{p} \end{bmatrix} = 0 \end{aligned} \tag{12}$$

3.2. SQP Method

We search the parameter value p satisfying the KKT condition by the SQP method for Eq.(11). At each iteration, we solve a quadratic programming (QP) problem

$$\begin{aligned} &\min_{\Delta p} \ \left\{ \frac{1}{2}\Delta p^T \nabla^2 f(p) \Delta p + \nabla f(p)^T \Delta p \right\} \\ &\text{s.t.} \quad -p+\underline{p}-\Delta p \le 0, \quad p-\overline{p}+\Delta p \le 0 \end{aligned} \tag{13}$$

to determine the direction vector Δp, where $\nabla^2 f(p)$ is equal to the Hesse matrix $\nabla_p{}^2 L(p,\mu)$ of the Lagrange function $L(p,\mu)$ with respect to p. When the solution Δp of Eq.(13) is 0, the design parameter vector p satisfies the KKT condition Eq.(12) and it is the optimal solution of Eq.(10). We employ the Goldfarb-Idnani method [6] to solve the QP problem Eq.(13).

In the SQP method for Eq.(11), computation of $\nabla^2 f(p)$ at each iteration is inefficient. Therefore, we employ the Powell's modified Broyden-Fletcher-Goldfarb-Shanno update [6]

$$\mathcal{B}_k = \mathcal{B}_{k-1} - \frac{\mathcal{B}_{k-1} s_k {s_k}^T \mathcal{B}_{k-1}}{{s_k}^T \mathcal{B}_{k-1} s_k} + \frac{\psi_k {\psi_k}^T}{{s_k}^T \psi_k} \tag{14}$$

$$\begin{aligned} s_k &= p_k - p_{k-1} \\ \chi_k &= \nabla f(p_k) - \nabla f(p_{k-1}) \\ \psi_k &= \phi_k \chi_k + (1-\phi_k)\mathcal{B}_{k-1} s_k \\ \phi_k &= \begin{cases} 1, & {s_k}^T \chi_k \ge 0.1 {s_k}^T \mathcal{B}_{k-1} s_k \\ \dfrac{0.9 {s_k}^T \mathcal{B}_{k-1} s_k}{{s_k}^T (\mathcal{B}_{k-1} s_k - \chi_k)}, & {s_k}^T \chi_k < 0.1 {s_k}^T \mathcal{B}_{k-1} s_k \end{cases} \end{aligned}$$

which gives an approximation $\mathcal{B}_k$ of $\nabla^2 f(p)$, where the subscript k means the k-th iteration. As the initial values, we assume that $\mathcal{B}_0$ is the unit matrix and $p_0 = 0$.

We note that we need $\nabla f(p)$ to compute Eqs.(13) and (14), but it cannot be computed analytically. Therefore, we compute the gradient $\nabla f(p)$ numerically by the difference of $f(p)$ for small changes of the node displacement vector p.

To apply the SQP method practically, we need to determine the step size in the direction Δp, which represents the size of variations of design parameters. If it is too small, a lot of iterations are needed to reach the optimal solution. If it is too large, the optimal solution may be missed. Therefore, the step size should be large, but has to decrease the objective function and satisfy the constraint.

In this paper, to find such a suitable step size, we adopt the idea of Armijo [6]. That is, we employ the ℓ_1 exact penalty function

$$P(p;\eta) = f(p) + \eta \sum_i \max(0,\, -p_{ki} + \underline{p}_i,\, p_{ki} - \overline{p}_i) \tag{15}$$

as the merit function, where η is a positive number and p_{ki}, $\underline{p}_i$, $\overline{p}_i$ are the i-th components of p_k, $\underline{p}$, $\overline{p}$, respectively. Then, we introduce the inequality

$$P(p_k + \delta_k \Delta p_k; \eta) \le P(p_k;\eta) - \xi \delta_k \Delta {p_k}^T \mathcal{B}_{k-1} \Delta p_k \tag{16}$$

to determine a large step size δ_k satisfying this inequality, where ξ is a constant such that $0 \le \xi \le 1$. Using this δ_k, we determine the design parameter vector p_{k+1} by

$$p_{k+1} = p_k + \delta_k \Delta p_k. \tag{17}$$

4. FEM MODELING

To illustrate the FEM modeling of the structure, we consider a five-stories Rahmen structure. Fig.1 shows the nominal shape of the structure, which is taken as the initial shape in the iterative optimization. We consider deformation of the structure by changing the node positions. The structure is fixed at nodes N1∼N4 on the floor in all directions. Then, the displacement vector q is composed of the translations x, y, z and rotations θ_x, θ_y, θ_z of nodes N5∼N24. The directions x, y, z are indicated in Fig.1.

We assume that the cross-sections of frame members are all square with the same size and denote the area by a. In this paper, we do not change the cross-sectional areas of frame members. Then, the total mass matrix $M(p)$ and total stiffness matrix $K(p)$ are described as

$$M(p) = W_c W_p(p) \hat{M}(p) W_p^{-1}(p) W_c^T \tag{18}$$

$$K(p) = W_c W_p(p) \hat{K}(p) W_p^{-1}(p) W_c^T \tag{19}$$

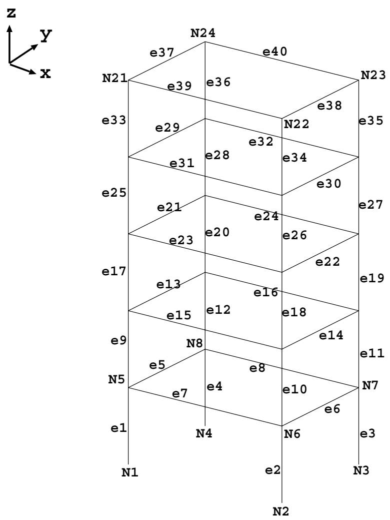


Fig. 1. Frame structure

where

$$\hat{M}(p) = \mathrm{diag}\{m_1,\ m_1,\ m_2,\ m_2,\ \cdots,\ m_{40},\ m_{40}\}$$
$$m_i = \frac{\rho \ell_i(p)}{2}\mathrm{diag}\{a,\ a,\ a,\ \frac{a^2}{6},\ \frac{a^2}{12},\ \frac{a^2}{12}\} \tag{20}$$

and

$$\hat{K}(p) = \mathrm{diag}\{\kappa_1,\ \kappa_2,\ \cdots,\ \kappa_{40}\} \tag{21}$$

$$\kappa_i =$$
$$\begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & -k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & k_5 & 0 & -k_2 & 0 & 0 & 0 & k_5 \\ 0 & 0 & k_2 & 0 & k_5 & 0 & 0 & 0 & -k_2 & 0 & k_5 & 0 \\ 0 & 0 & 0 & k_3 & 0 & 0 & 0 & 0 & 0 & -k_3 & 0 & 0 \\ 0 & 0 & k_5 & 0 & 2k_4 & 0 & 0 & 0 & -k_5 & 0 & k_4 & 0 \\ 0 & k_5 & 0 & 0 & 0 & 2k_4 & 0 & -k_5 & 0 & 0 & 0 & k_4 \\ -k_1 & 0 & 0 & 0 & 0 & 0 & k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 & 0 & -k_5 & 0 & k_2 & 0 & 0 & 0 & -k_5 \\ 0 & 0 & -k_2 & 0 & -k_5 & 0 & 0 & 0 & k_2 & 0 & -k_5 & 0 \\ 0 & 0 & 0 & -k_3 & 0 & 0 & 0 & 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & k_5 & 0 & k_4 & 0 & 0 & 0 & -k_5 & 0 & 2k_4 & 0 \\ 0 & k_5 & 0 & 0 & 0 & k_4 & 0 & -k_5 & 0 & 0 & 0 & 2k_4 \end{bmatrix}$$

$$k_1 = \frac{ea}{\ell_i(p)},\quad k_2 = \frac{ea^2}{\ell_i^3(p)},\quad k_3 = \frac{ea^2}{6\ell_i(p)(1+\nu)}$$
$$k_4 = \frac{ea^2}{6\ell_i(p)},\quad k_5 = \frac{ea^2}{2\ell_i^2(p)}$$

are the member-level mass and stiffness matrices both in the local coordinate system. In Eqs.(20) and (21), ρ is the density of the material of frame members, $\ell_i(p)$ $(i = 1, 2, \cdots, 40)$ is the length of the i-th member, e is the Young's modulus, and ν is the Poisson's ratio. In (18) and (19), $W_p(p)$ is the transformation matrix from the local coordinate system to the global coordinate system, and W_c is the constraint matrix defined by the connections of members. We note that the length $\ell_i(p)$ of members and the transformation matrix $W_p(p)$ are dependent on the node positions.

5. EXAMPLE

In this section, we present examples to demonstrate the effectiveness of the proposed design method.

5.1. Design Condition

We consider the nominal frame structure depicted in Fig.1 with the following non-dimensional constants. The lengths of beams in the x direction, beams in the y direction, and columns are 500, 300 and 200, respectively, which define the initial node positions of the iterative structural design. The cross-sectional areas are all 400. The density ρ is 7.8×10^{-6}, the Young's modulus e is 2.1×10^5, and Poisson ratio ν is 0.33. The coefficients α and β of the proportional damping matrix D in Eq.(2) are supposed to be 10^{-6}.

We design the structure in the following situation. The disturbance input excites node N5 in the $y = x$ direction as shown in Fig.2. Displacements of nodes N21∼N24 in the x, y and z directions are the outputs, which we want to suppress. We deal with a structure in symmetry in the x and y directions. The heights of the nodes on the same story are equal. We do not change the positions of nodes N1∼N4

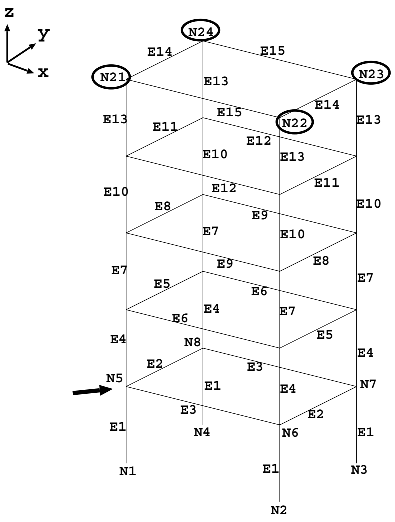


Fig. 2. Input and output points

and N21∼N24. The lower bounds of the position changes of nodes N5∼N20 from the nominal ones in the x, y and z directions are -100, -60 and -40, and the corresponding upper bounds are 100, 60 and 40, respectively.

5.2. Design Result

Let us first consider the case of the coefficients $\gamma_1 = \gamma_2 = 0.5$ in Eq.(10), which we call Case 1. Fig.3 illustrates the design result, where the dash-dotted lines show the nominal structure and the solid lines show the optimized structure. The total weight of the nominal structure is 37.4 and the optimized structure is 35.6. The reduction ratio is 4.8%.

Fig.4 gives the frequency responses of the structures. The H_∞ norm of the nominal structure is 2.92 (9.31dB) at 149.2 (rad/sec) and that of the optimized structure is 0.21 (-13.46dB) at 440.8(rad/sec). The reduction ratio is 92.7%. We have achieved a significant improvement.

Design results depend on the coefficients in Eq.(10). We consider two other cases, that is, Case 2: $\gamma_1 = 0.1$, $\gamma_2 = 0.9$ and Case 3: $\gamma_1 = 0.9$, $\gamma_2 = 0.1$. Figs.5 and 7 illustrate the design results of Cases 2 and 3, respectively and Figs.6 and 8 give the corresponding frequency responses. The total weight of the optimized structure of Case 2 is 32.8 and that of Case 3 is 35.9. The reduction ratios are 12.4 % and 4.2%, respectively. The H_∞ norm of the optimized structure of Case 2 is 0.50 (-5.94dB) at 188.6 (rad/sec) and that of Case 3 is 0.19 (-14.51dB) at 434.1(rad/sec). The reduction ratios are 82.7 % and 93.6%, respectively.

We see that as the coefficient γ_1 for the H_∞ norm increases relatively to the coefficient γ_2 for the total weight in Eq.(10), the H_∞ norm decreases and the total weight increases in the optimization. Therefore, the H_∞ norm and the total weight of any case are not simultaneously better or worse than other cases. These solutions are actually the Pareto optimal solutions.

The optimized shapes also depend on the choice of γ_1 and γ_2. However, there are two common features. First, the first and the second stories are lower than the nominal ones. It

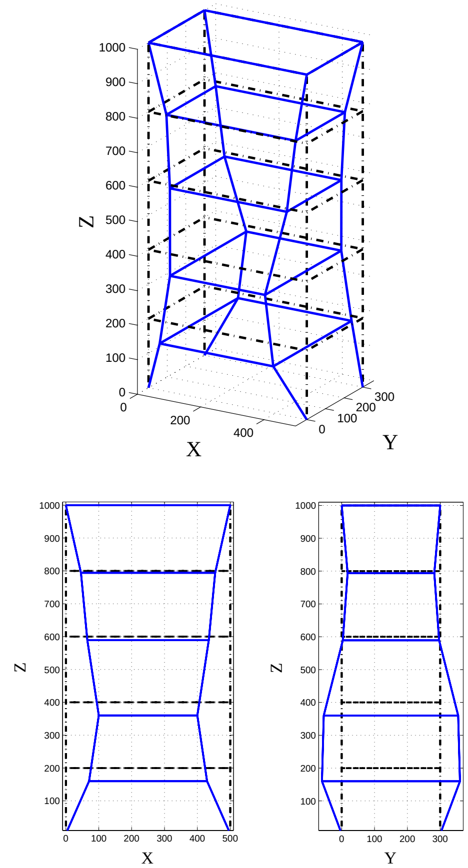


Fig. 3. Design Result: Case1

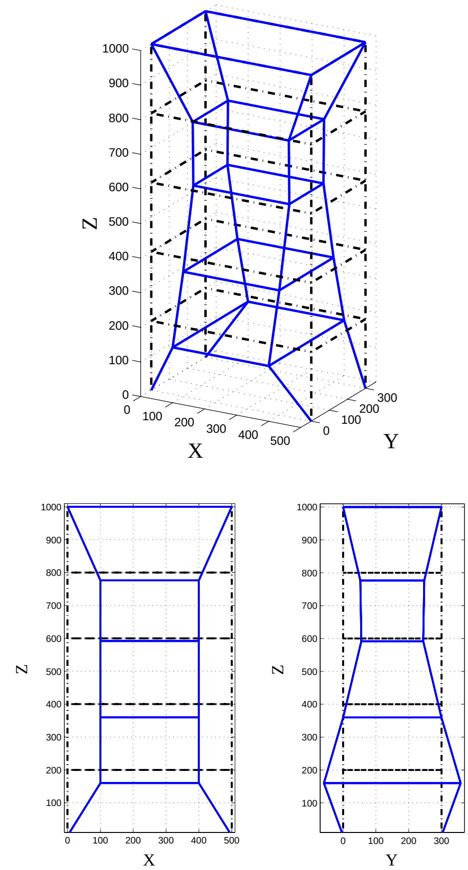


Fig. 5. Design Result: Case2

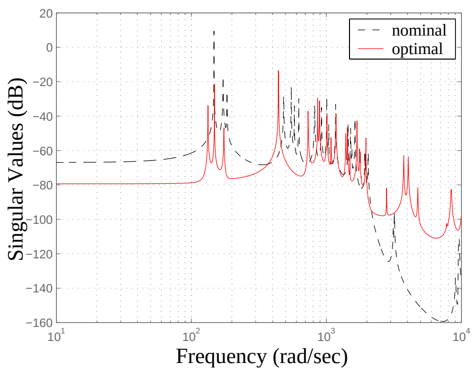


Fig. 4. Frequency Responses: Case1

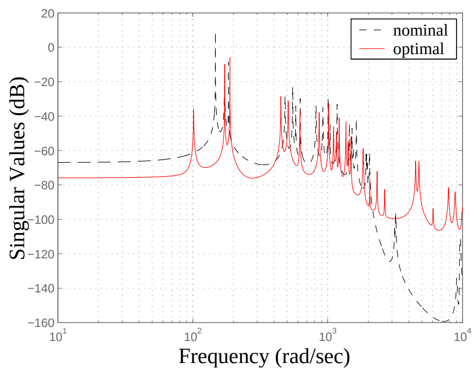


Fig. 6. Frequency Responses: Case2

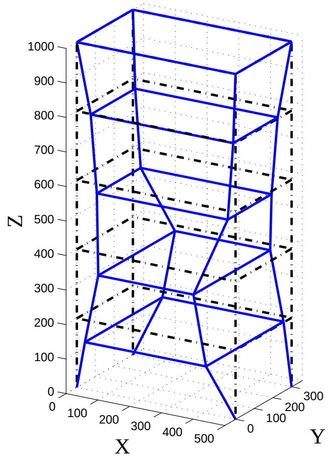
may be supposed that such a shape increases the stiffness at the input point. Second, on the first story, the beams in the x direction are shorter than those of the y direction. This shape also may increase the stiffness by compensating the shape defined by N1∼N4 on the fixed floor where beams in the x direction are longer than those of the y direction.

6. CONCLUSION

We have proposed two-objective design of frame structures for vibration suppression and weight reduction. The design parameters are the node positions of the structure. The Pareto optimal solutions are computed by dealing with a number of linear combinations of the H_∞ norm of the transfer function from disturbance inputs to displacement outputs and the total weight.

We have employed the SQP method to solve the optimization problem and demonstrated the effectiveness of the proposed design method by numerical examples.

To reduce the computation time for the H_∞ norm of high-dimensional systems, we have proposed elimination of modes, which appear insignificantly in the input-output property. We have applied the balanced realization technique for model reduction as well.



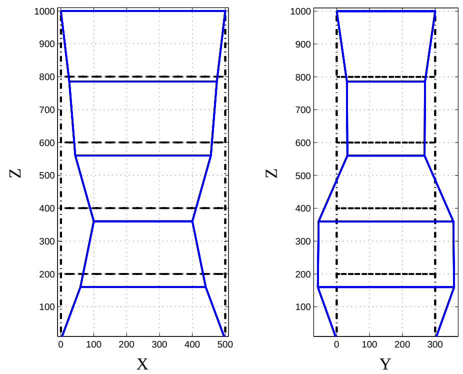


Fig. 7. Design Result: Case3

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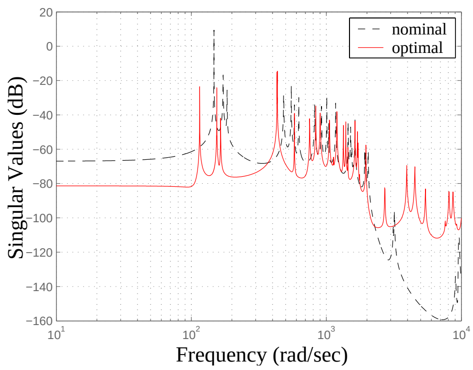

Fig. 8. Frequency Responses: Case3

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