# Robust Optimal Control of Robot Manipulators with a Weighting Matrix Determination Algorithm

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**Abstract**: A robust optimal control design is proposed in this study for rigid robotic systems under the unknown load and the other uncertainties. The uncertainties are quadratically bounded for some positive definite matrix. Iterative method finding the Q weighting matrix is shown. Computer simulations have been done for a weight-lifting operation of a two-link manipulator and the result of the simulation shows that the proposed algorithm is very effective for a robust control of robotic systems.

**Keywords**: optimal control, uncertainties, weighting matrix, iterative method, robust control

## 1. INTRODUCTION

The motion control of a robot manipulator has received a great deal of attention in the past decade. Many approaches have been introduced to treat this control problem[1]. Because of the unknown load placed on the manipulator and the other uncertainties in the manipulator dynamics, adaptive control approaches and robust control approaches have been proposed to attenuate these uncertainties. Johansson[2] proposed explicit solutions to the Hamilton-Jacobi equation for optimal control of rigid body motion and designed adaptive control for self-optimization to solve the case of unknown or uncertain system parameters. Chen[3] proposed a mixed $H_2/H_\infty$ control design for tracking of rigid robotic systems under parameter perturbations and external disturbances. And Lin[4] translateed the robust control problem into the optimal control problem, where the uncertainties were reflected in the performance index.

This study has been done, based on Lin's work. The dynamics of a robot manipulator is to be written as the state space description by the definition of the state variables, control inputs and uncertainties. We define the uncertainties as the function of the state variables. To translate the robust control problem into the optimal control problem, we have to find a quadratically bounded matrix for the uncertainties. We simplify the problem to find the matrix by its definition as the product of a scalar and an identity matrix. And we define the state variables and the uncertainties as vectors on the finite time interval. Then we search the largest value of the square of the uncertainties divided by the square of the state variables on each time interval. The largest value is selected as the weighting value of the weighting matrix of the state variables in the cost function to be minimized. The optimal control inputs that minimize the cost function are obtained by solving the algebraic Riccati equation. And the control inputs are applied to the manipulator and the states of the system are changed. So the uncertainties are changed and we get the new largest value from the vectors of the uncertainties and state variables changed. The algorithm of searching the largest value includes the repeated routine stopped when the system gives us the same response.

We simulate the proposed algorithm for a weight-lifting operation of a two-link manipulator. We can see that the control is very robust with respect to the change of the load.

## 2. MANIPULATOR DYNAMICS

The dynamics of a robot manipulator is well understood and is given by

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \quad (1)$$

The position coordinates $q$ with associated velocities $\dot{q}$ and accelerations $\ddot{q}$ are controlled with the driving forces $\tau$. The moment of inertia $M(q)$, the Coriolis, centripetal, and frictional forces $C(q,\dot{q})\dot{q}$, and the gravitational forces $G(q)$ all vary along the trajectories. For simplicity, we denote

$$N(q,\dot{q}) = C(q,\dot{q})\dot{q} + G(q)\,. \quad (2)$$

There are uncertainties in $M(q)$ and $N(q,\dot{q})$ due to unknown load on the manipulator and unmodeled frictions. We assume the following bounds on the uncertainties:

1) There exists $M_o(q)$ such that $M(q) \le M_o(q)$.
2) There exists $N_o(q,\dot{q})$ such that $\|N(q,\dot{q})\| \le \|N_o(q,\dot{q})\|$.

From the dynamics of a robot manipulator, we have

$$\begin{aligned}\ddot{q} &= M^{-1}(\tau - N)\\ &= M^{-1}(\tau - N) - M^{-1}N_o + M^{-1}N_o\\ &= M^{-1}(\tau - N_o) + M^{-1}(N_o - N)\\ &= M^{-1}M_oM_o^{-1}(\tau - N_o) + M^{-1}M_oM_o^{-1}(N_o - N)\end{aligned} \quad (3)$$

where $M$, $N$ are shorter notation of $M(q)$, $N(q,\dot{q})$, and $M_o$, $N_o$ are shorter notation of $M_o(q)$, $N_o(q,\dot{q})$, respectively.

Let us define the control input $u$ and the uncertainty $w$ as

$$u = M_o^{-1}(\tau - N_o)\,,\quad w = M_o^{-1}(N_o - N) \quad (4)$$

The joint accelerations $\ddot{q}$ are given by

$$\ddot{q} = M^{-1}M_o u + M^{-1}M_o w \quad (5)$$

Define the state variables to be

$$x = \begin{bmatrix}\dot{q}\\ q\end{bmatrix} \quad (6)$$

Then, the state equation is given by

$$\dot{x} = Ax + Bu + Bw \quad (7)$$

where

$$A = \begin{bmatrix}0 & 0\\ I & 0\end{bmatrix},\quad B = M^{-1}M_o\begin{bmatrix}I\\ 0\end{bmatrix}. \quad (8)$$

## 3. OPTIMAL CONTROL APPROACH

The robust control problem can be translateed into the optimal control problem and if the solution to the optimal control problem exists, then it is a solution to the robust control problem[4].

Our goal is to solve the following robust control problem.

1) Robust Control Problem: Find a feedback control law such that the closed-loop system as $\dot{x} = Ax + Bu + Bw$, is globally asymptotically stable for all uncertainties $w$ satisfying the condition that there exists a nonnegative function $w_o$ such that $\|w\| \le w_o$.

We would like to translate this robust control problem into the following optimal control problem.

2) Optimal Control Problem: For the following system as $\dot{x} = Ax + Bu$, find a feedback control law that minimizes the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left(w_o^T w_o + x^T Qx + u^T Ru\right)dt \quad (9)$$

To translate the robust control problem into the optimal control problem, we need to assume that the uncertainty $w$ satisfies the following condition:

$$w^T Rw < x^T Q_w x \quad (10)$$

for some positive definite matrix $Q_w$.

Then the optimal control problem reduces to the following linear quadratic regulator(LQR) problem: For the system as $\dot{x} = Ax + Bu$, find a feedback control law that minimizes the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right)dt \quad (11)$$

The Hamiltonian is

$$H = \frac{1}{2}\left(x^T Q_w x + x^T Qx + u^T Ru\right) + J_x^{*T}\left(Ax + Bu\right) \quad (12)$$

where the minimum cost function $J^*$ is

$$J^* = \min_u \left\{\frac{1}{2}\int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right)dt\right\} \quad (13)$$

The $u = u^*$ for which $H$ has its minimum value is obtained from the partial derivatives with respect to $u$.

$$\frac{\partial H}{\partial u} = Ru + B^T J_x^* = 0\ , \quad J_x^{*T} B = -\left(u^*\right)^T R \quad (14)$$

The $Hamilton - Jacobi$ equation gives us such that

$$H^* = \frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right) + J_x^{*T}\left(Ax + Bu^*\right) = 0$$

and $J_x^{*T}\left(Ax + Bu^*\right) = -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right)$.

Let us define the Lyapunov function candidate $V$ as the minimum cost function $J^*$

$$V = J^* = \min_u \left\{\frac{1}{2}\int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right)dt\right\} \quad (15)$$

Solving Eq. (15) gives us the following Eqations such as

$$V_x^T B = -\left(u^*\right)^T R \quad (16)$$

$$V_x^T\left(Ax + Bu^*\right) = -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right) \quad (17)$$

To show $\dot{V} = dV/dt < 0$, we have

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x}\frac{dx}{dt} \\ &= V_x^T\left(Ax + Bu^* + Bw\right) \\ &= V_x^T\left(Ax + Bu^*\right) + V_x^T Bw \\ &= -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right) - \left(u^*\right)^T Rw \\ &= -\frac{1}{2}\left(x^T Q_w x - w^T Rw\right) - \frac{1}{2}x^T Qx - \frac{1}{2}\left(u^* + w\right)^T R\left(u^* + w\right)\end{aligned} \quad (18)$$

By the assumption $x^T Q_w x - w^T Rw > 0$ shown in Eq. (10), the Lyapunov function derivative is negative definite. Thus, the condition of the Lyapunov global asymptotic stability theorem is satisfied. The solution can be obtained by solving the following algebraic Riccati equation:

$$0 = Q_w + Q - PBR^{-1}B^T P + PA + A^T P \quad (19)$$

and the optimal control is given by

$$u^* = -R^{-1}B^T Px \quad (20)$$

## 4. CHOICE OF THE WEIGHTING MATRICES

We need to select the weighting matrices $Q$, $R$ and $Q_w$ to find the optimal control $u^*$. The weight matrix $Q_w$ can be selected from the assumption of the uncertainties $w$.

Let us define $R = rI$ and $Q_w = q_w I$, then $w^T Rw < x^T Q_w x$ is given by

$$\frac{\|w\|^2}{\|x\|^2} < \frac{q_w}{r} = \gamma \quad (21)$$

To get the weighting value $\gamma$ that is satisfied the above condition in finite time interval [0, N], let us define the state variables $x$ and the uncertainties $w$ as follows:

$$x_k = \left(x_k(0), x_k(1), ..., x_k(N)\right) \quad (22)$$

$$w_k = \left(w_k(0), w_k(1), ..., w_k(N)\right) \quad (23)$$

where the subscript $k$ denotes the $k$ th trial, the weighting value $\gamma_k$ in the $k$ th trial is given by the following $l_\infty - norm$ such as

$$\gamma_k = \|\Gamma_k\|_\infty = \max_{0 \le i \le N}\left|\Gamma_k(i)\right| \quad (24)$$

where

$$\Gamma_k = \left(\Gamma_k(0), \Gamma_k(1), ..., \Gamma_k(N)\right) \quad (25)$$

$$\Gamma_k(i) = \frac{\|w_k(i)\|^2}{\|x_k(i)\|^2}\ , \quad i = 0, ..., N \quad (26)$$

To avoid $\|x_k(i)\|^2 = 0$, dummy variable $\delta$ is used as follows:

$$\Gamma_k{}'(i) = \frac{\|w_k(i)\|^2}{\|x_k(i)\|^2 + \delta}\ , \quad \delta \approx 0 \quad (27)$$

Now then, we have

$$\Gamma_k(i) = \frac{\|w_k(i)\|^2 \Gamma_k'}{\|w_k(i)\|^2 - \delta\Gamma_k'} \quad (28)$$

The computation to find the weighting value $\gamma$ is repeated until the following convergence criterion is satisfied such as

$$\| \gamma_k - \gamma_{k-1} \| < \varepsilon \quad (29)$$

where $\varepsilon$ is a given error requirement.

## 5. EXMAPLE

We illustrate the proposed optimal control approach by an example of a two-link robot manipulator in Fig. 1, with point masses $m_1$, $m_2$ (*kg*), lengths $l_1$, $l_2$ (*m*), angular positions $q_1$, $q_2$ (*rad*), and torques $\tau_1$, $\tau_2$ (*Nm*). The parameters for the equation of motion are

$$M(q) = \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{pmatrix}$$

$$C(q,\dot{q}) = m_2 l_1 l_2 (c_1 s_2 - s_1 c_2) \begin{pmatrix} 0 & -\dot{q}_2 \\ +\dot{q}_1 & 0 \end{pmatrix}$$

$$G(q) = \begin{pmatrix} -(m_1+m_2) l_1 g s_1 \\ -m_2 l_2 g s_2 \end{pmatrix}$$

and the short-hand notations $c_1 = \cos(q_1)$, $s_1 = \sin(q_1)$, $c_2 = \cos(q_2)$, $s_2 = \sin(q_2)$ are used.

For the convenience of simulation, the nominal parameters of the robotic system are given as $m_1 = 1(kg)$, $m_2 = 10(kg)$, $l_1 = 1(m)$, $l_2 = 1(m)$ and the initial values $q_1 = q_2 = \pi/2$ (rad), $\dot{q}_1 = \dot{q}_2 = 0$. The reference values are $q_r = 0$, $\dot{q}_r = 0$. $M(q)$ and $N(q,\dot{q})$ are the function of $q$

| *k* | *t* | *gamma* | *gamma'* | *xx(5001)* | *xxSUM* | *JSUM* |
|---|---|---|---|---|---|---|
| 000 | 0.000 | 00000.000000 | 00000.000000 | 00.000000 | 009836.282035 | 000025090.563501 |
| 001 | 2.894 | 35214.557504 | 17607.278752 | 00.000000 | 002475.407387 | 043729280.111337 |
| 002 | 0.000 | 00004.865442 | 08806.072097 | 00.000000 | 002478.113223 | 021920072.912788 |
| 003 | 0.000 | 00004.865442 | 04405.468770 | 00.000000 | 002482.352965 | 011005327.965142 |
| 004 | 0.000 | 00004.865442 | 02205.167106 | 00.000000 | 002489.043725 | 005540294.228730 |
| 005 | 0.000 | 00004.865442 | 01105.016274 | 00.000000 | 002499.909716 | 002802513.246557 |
| 006 | 0.000 | 00004.865442 | 00554.940858 | 00.000000 | 002518.140607 | 001430139.688169 |
| 007 | 0.058 | 00005.000576 | 00279.970717 | 00.000000 | 002549.343326 | 000741751.333341 |
| 008 | 0.334 | 00006.210467 | 00143.090592 | 00.000000 | 002602.273032 | 000397327.098876 |
| 009 | 0.331 | 00007.913236 | 00075.501914 | 00.000000 | 002687.932188 | 000225942.496634 |
| 010 | 0.341 | 00009.662743 | 00042.582328 | 00.000000 | 002812.639578 | 000141548.269939 |
| 011 | 0.500 | 00011.318062 | 00026.950195 | 00.000000 | 002961.734013 | 000100917.064449 |
| 012 | 0.517 | 00027.498686 | 00027.224440 | 00.000000 | 002957.832099 | 000101635.663050 |
| 013 | 0.516 | 00027.005438 | 00027.114939 | 00.000000 | 002959.381718 | 000101348.772558 |
| 014 | 0.516 | 00027.199120 | 00027.157030 | 00.000000 | 002958.784768 | 000101459.053461 |
| 015 | 0.516 | 00027.124580 | 00027.140805 | 00.000000 | 002959.014684 | 000101416.543630 |
| 016 | 0.516 | 00027.153300 | 00027.147052 | 00.000000 | 002958.926124 | 000101432.912580 |
| 017 | 0.516 | 00027.142239 | 00027.144646 | 00.000000 | 002958.960235 | 000101426.606936 |
| 018 | 0.516 | 00027.146500 | 00027.145573 | 00.000000 | 002958.947096 | 000101429.035614 |
| 019 | 0.516 | 00027.144859 | 00027.145216 | 00.000000 | 002958.952157 | 000101428.100129 |
| 020 | 0.516 | 00027.145491 | 00027.145353 | 00.000000 | 002958.950208 | 000101428.460454 |
| 021 | 0.516 | 00027.145247 | 00027.145300 | 00.000000 | 002958.950958 | 000101428.321665 |
| 022 | 0.516 | 00027.145341 | 00027.145321 | 00.000000 | 002958.950669 | 000101428.375123 |
| 023 | 0.516 | 00027.145305 | 00027.145313 | 00.000000 | 002958.950781 | 000101428.354532 |
| 024 | 0.516 | 00027.145319 | 00027.145316 | 00.000000 | 002958.950738 | 000101428.362463 |
| 025 | 0.516 | 00027.145313 | 00027.145315 | 00.000000 | 002958.950754 | 000101428.359408 |
| 026 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950748 | 000101428.360585 |
| 027 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360132 |
| 028 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950749 | 000101428.360306 |
| 029 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360239 |
| 030 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360265 |
| 031 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360255 |
| 032 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360259 |
| 033 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360257 |
| 034 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 035 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 036 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 037 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 038 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 039 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |
| 040 | 0.516 | 00027.145315 | 00027.145315 | 00.000000 | 002958.950750 | 000101428.360258 |

Table. 1 The result of Iterative Method for $m_L = 0.$, $\gamma_k' = (\gamma_{k-1}' + \gamma_k)/2$

and $\dot{q}$. So, they are changed each moment according to the motion of the manipulator and it is not easy to get the boundary. But in the regulator problem of weight-lifting operation, if it is assumed that the dynamics of the manipulator includes the uncertainty of the mass of the unknown load only, it is not difficult to calculate $M(q)$ and $N(q,\dot{q})$ when the mass of the unknown load is the maximum value. They are selected as $M_o(q)$ and $N_o(q,\dot{q})$ and used in simulations. If the maximum value is 10(kg), $M_o$, $C_o$, $G_o$ are given by

$$M_o(q)=\begin{pmatrix} 21 & 20(s_1s_2+c_1c_2) \\ 20(s_1s_2+c_1c_2) & 20 \end{pmatrix}$$

$$C_o(q,\dot{q})=20(c_1s_2-s_1c_2)\begin{pmatrix} 0 & -\dot{q}_2 \\ +\dot{q}_1 & 0 \end{pmatrix}$$

$$G_o(q)=\begin{pmatrix} -21gs_1 \\ -20gs_2 \end{pmatrix}.$$

Iteration method is initialized by the weighting matrices $Q=I$, $R=I$ and the weighting value $\gamma=0$. To get the converged weighting value $\gamma$, we use $\gamma'_k=(\gamma'_{k-1}+\gamma_k)/2$ instead of $\gamma_k$ for the $k$ th trial. The result of iterative method for the mass of the load $m_L=0$ is shown in Table 1. In the $k$ th trial, the largest weighting value $\gamma_k$, *gamma* on finite time interval, and $\gamma'_k$, *gamma'*, the very moment $t$, the squared value of the state variable on the final time interval $\|x_k(N)\|^2$, *xx(5001)*, the sum of the squared value of the state variable on the finite time interval $\sum_{i=0}^{N}\|x_k(i)\|^2$, *xxSUM*, and the cost function $J$, *JSUM* are listed in Table 1. Since the 34 th trial, the largest weighting value $\gamma$ and the very moment $t$ converge to $\gamma=27.145315$, $t=0.516$ and the system is to be steady state.

The weighting matrix selected from the proposed iterative method $Q_w=27.145315I$ and the weighting matrices $Q=I$, $R=I$ are used to solve the algebraic Riccati equation. The simulation results for γ=0 and γ=27.145315 are shown in Figs. 2-7. Fig. 2 shows the joint position q1, joint velocity q1dot, applied torque τ1 for $m_L=0$. Fig. 3 shows the joint position q2, joint velocity q2dot, applied torque τ2 for $m_L=0$. Figs. 4-5 are the results for $m_L=5$ and Figs. 6-7 are the results for $m_L=10$. From the figures, we can see that the control is very robust with respect to the change in the load.

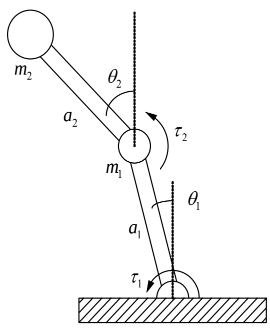


Fig. 1 A two-link manipulator with masses $m_1$ and $m_2$.

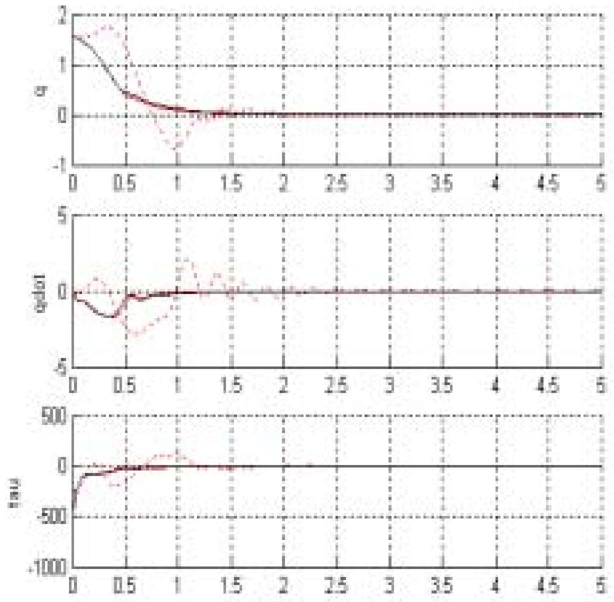


Fig. 2 Response for the mass of the load $m_L$=0(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q1, middle graph shows the joint velocity q1dot and lower graph shows the applied torque τ1, respectively. All graphs versus time(s).

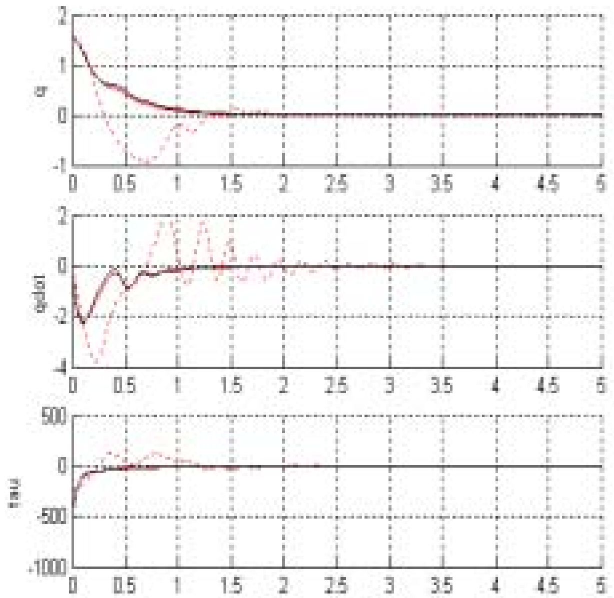


Fig. 3 Response for the mass of the load $m_L$=0(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q2, middle graph shows the joint velocity q2dot and lower graph shows the applied torque τ2, respectively. All graphs versus time(s).

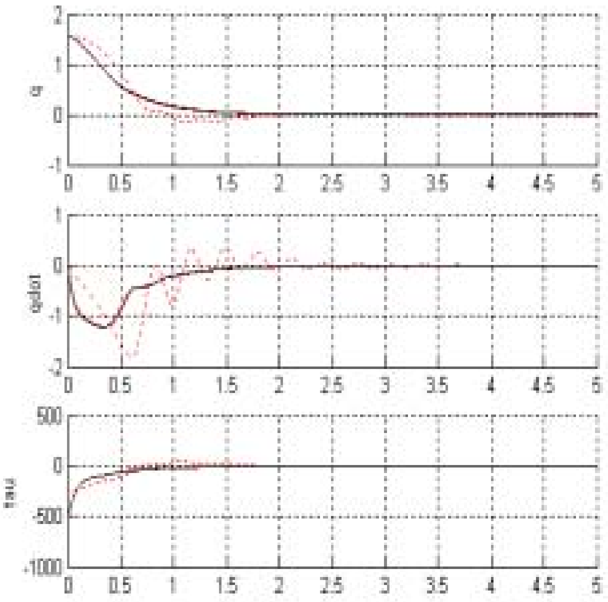


Fig. 4 Response for the mass of the load $m_L$=5(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q1, middle graph shows the joint velocity q1dot and lower graph shows the applied torque τ1, respectively. All graphs versus time(s).

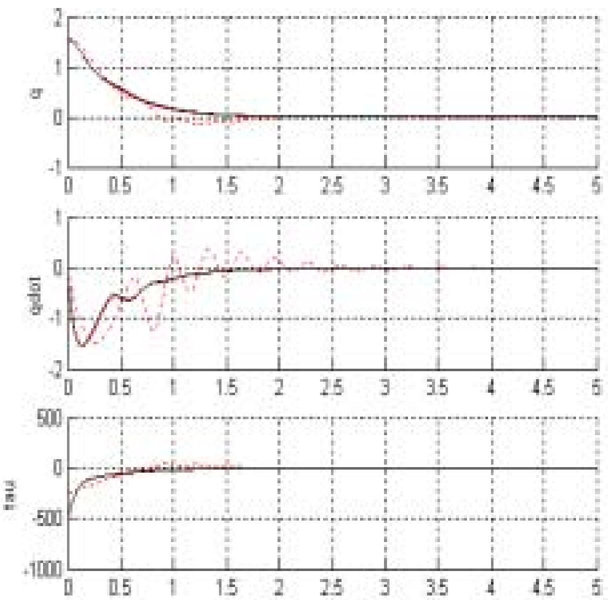


Fig. 5 Response for the mass of the load $m_L$=5(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q2, middle graph shows the joint velocity q2dot and lower graph shows the applied torque τ2, respectively. All graphs versus time(s).

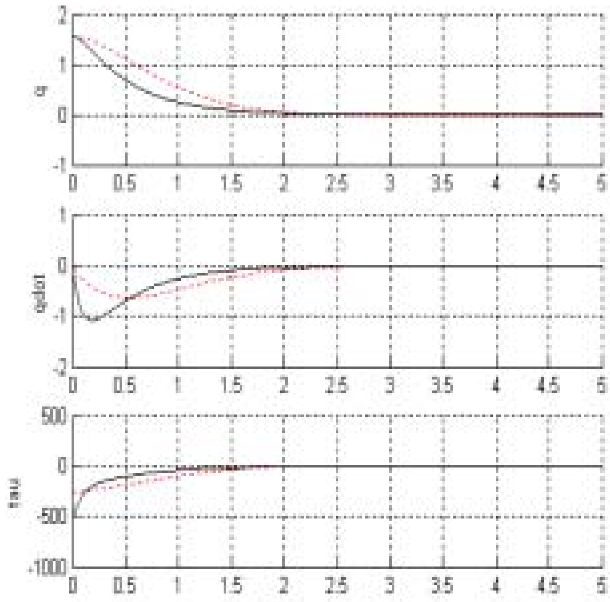


Fig. 6 Response for the mass of the load $m_L$=10(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q1, middle graph shows the joint velocity q1dot and lower graph shows the applied torque τ1, respectively. All graphs versus time(s).

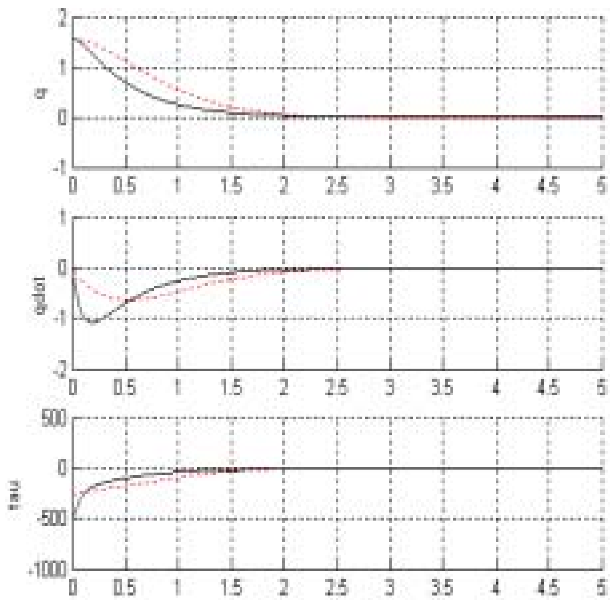


Fig. 7 Response for the mass of the load $m_L$=10(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
Upper graph shows the joint position q2, middle graph shows the joint velocity q2dot and lower graph shows the applied torque τ2, respectively. All graphs versus time(s).

## 6. CONCLUSION

We presented a robust optimal control of robot manipulators using the algorithm to choose the weighting matrix. The dynamics of a robot manipulator has been written as the state space description by the definition of the state variables, control inputs and uncertainties. To translate the robust control problem into the optimal control problem, we have to find a quadratically bounded matrix for the uncertainties. We simplified the problem to find the matrix by its definition as the product of a scalar and an identity matrix. We defined the state variables and the uncertainties as vectors on the finite time interval. We proposed an algorithm that searches the largest value of the uncertainties on the finite time interval iteratively. The weighting matrix selected by the proposed algorithm has been used in our simulations. Simulations have been done for a weight-lifting operation of a two-link manipulator and the result of the simulation shows that the proposed algorithm is very effective for a robust control of robotic systems.

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