A Globally Stabilizing Model Predictive Controller for Neutrally Stable Linear Systems with Input Constraints

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Abstract: MPC or model predictive control is representative of control methods which are able to handle physical constraints. Closed-loop stability can therefore be ensured only locally in the presence of constraints of this type. However, if the system is neutrally stable, and if the constraints are imposed only on the input, global aymptotic stability can be obtained; until recently, use of infinite horizons was thought to be inevitable in this case. A globally stabilizing finite-horizon MPC has lately been suggested for neutrally stable continuous-time systems using a non-quadratic terminal cost which consists of cubic as well as quadratic functions of the state. The idea originates from the so-called small gain control, where the global stability is proven using a non-quadratic Lyapunov function. The newly developed finite-horizon MPC employs the same form of Lyapunov function as the terminal cost, thereby leading to global asymptotic stability. A discrete-time version of this finite-horizon MPC is presented here. The proposed MPC algorithm is also coded using an SQP (Sequential Quadratic Programming) algorithm, and simulation results are given to show the effectiveness of the method.

Keywords: MPC, global asymptotic stability, non-quadratic cost, input constraints

1. Introduction

MPC or model predictive control is a receding horizon strategy, where the control is computed via an optimization procedure at every sampling instant. It is therefore possible to handle physical constraints on the input and/or state variables through the optimization[1]. Over the last decade, there have been many stability results on constrained MPC. Moreover, explicit solutions to constrained MPC are proposed recently[2], [3]. Theses results reduce on-line computational burden regarded as a main drawback of MPC and extend the applicability of MPC to faster plants as in electrical applications.

Particular attention is paid in this paper to input-constrained systems as almost all real processes are subject to actuator saturation. Generally, it is not possible to stabilize input-constrained plants globally. However, if the unconstrained part of the system is neutrally stable[1], then global stabilization can be achieved. A typical example is the so-called small gain control[4], [5], [6]; it is noted that the Lyapunov functions used for stability analysis are non-quadratic functions containing cubic as well as quadratic terms.

Global stabilization of input-constrained neutrally stable systems is also possible via MPC; see e.g. [7]. As in [7], use of infinite horizons is generally thought to be inevitable. However, infinite horizon MPC can cause trouble in practice. For implementation, the optimization problem should be reformulated as a finite horizon MPC with a variable horizon, and it is not possible to predetermine a finite upper bound on the horizon.

It is only fairly recently that globally stabilizing finite horizon MPC has been proposed for continuous-time neutrally stable systems[13]. This late achievement is based on two observations; firstly, the stability of an MPC system is mostly proved by showing that the terminal cost is a Lyapunov function[1], [8]. Secondly, the global stabilization of an input-constrained neutrally stable system can be achieved by using a non-quadratic Lyapunov function as mentioned above. By making use of these two facts, a new finite horizon MPC has been suggested in [13], where a non-quadratic Lyapunov function as in [4], [5], [6] is employed as the terminal cost, thereby guaranteeing the closed-loop stability. Here, we present a discrete-time version of this newly developed finite-horizon MPC in [13]. The proposed MPC algorithm is also coded using the SQP, and simulation results are given to show the effectiveness of the method.

2. An overview of MPC

Following [1], a brief summary on MPC is given in this section. Consider a discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k)$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and input, and (A, B) is assumed to be controllable. Defining

$$\mathbf{u}(k) = \{u(k|k), u(k+1|k), \cdots, u(k+N-1|k)\},$$

[1] All eigenvalues lie within the unit circle and those on the unit circle are simple.

the MPC law is obtained by minimizing with respect to $\mathbf{u}(k)$

$$\begin{aligned}&J_N(x(k),\mathbf{u}(k))\\&=\sum_{i=0}^{N-1} l(x(k+i|k),u(k+i|k))+V(x(k+N|k))\end{aligned}$$

subject to

$$\begin{aligned}&x(k+i+1|k)=Ax(k+i|k)+Bu(k+i|k),\ x(k|k)=x(k)\\&x(k+i+1|k)\in\mathcal{X},\quad u(k+i|k)\in\mathcal{U},\quad i\in[0,N-1]\\&x(k+N|k)\in\mathcal{X}_f\end{aligned}$$

where

$$\begin{aligned}&l(x(k+i|k),u(k+i|k))\\&=x(k+i|k)^TQx(k+i|k)+u(k+i|k)^TRu(k+i|k)\end{aligned} \quad (1)$$

with Q and R being positive definite, $V(x(k+N|k))$ is the terminal cost, the sets $\mathcal{U}$, $\mathcal{X}$ represent the input and state constraints, and $x(k+N|k)\in\mathcal{X}_f$ is the artificial terminal constraint employed for stability guarantees. Note that $V(x)$ and $\mathcal{X}_f$ are chosen such that $V(x)$ is a control Lyapunov function in $\mathcal{X}_f$. The entire procedure is repeated at each sampling instant, i.e.

$$u(k)=u^*(k|k)$$

where $u^*(k+i|k)$ is the optimal value of $u(k+i|k)$. The stability properties of the resulting closed-loop are summarized below.

Theorem 1: [1] For some local controller $k_f: X_f\rightarrow\mathbb{R}$, suppose the following:

A1. $\mathcal{X}_f$ is closed and $0\in X_f$;
A2. $k_f(x)\in\mathcal{U},\ \forall x\in X_f$;
A3. $\mathcal{X}_f$ is invariant, i.e, $Ax+Bk_f(x)\in X_f,\ \forall x\in X_f$;
A4. $V(Ax+Bk_f(x))-V(x)+l(x,k_f(x))\le 0,\ \forall x\in X_f$.

Then the optimization problem is guaranteed to be feasible for all $k\ge 0$ as long as the initial state $x(0)$ can be steerable to X_f in N steps or less (i.e. the problem is feasible initially). In addition, the optimal cost $J^*(k)$ at time k, i.e. the minimal value of $J_N(x(k),\mathbf{u}(k))$ satisfies

$$J^*(k+1)-J^*(k)+l(x(k),u^*(k|k))\le 0,$$

thereby ensuring asymptotic stability of the closed-loop.

Outline of proof: Suppose that

$$\mathbf{u}^*(k)=\{u^*(k|k),\cdots,u^*(k+N-1|k)\}$$

is the optimal (and thus feasible) $\mathbf{u}(k)$ obtained at time k, and consider

$$\tilde{\mathbf{u}}(k+1)=\{u^*(k+1|k),\cdots,u^*(k+N-1|k),k_f(x^*(k+N|k))\}$$

where

$$x^*(k+i+1|k)=Ax^*(k+i|k)+Bu^*(k+i|k),\ i\in[0,N-1].$$

Note that $x^*(k+1|k)=x(k+1)$ as $u^*(k|k)=u(k)$ and that $x^*(k+N|k)\in\mathcal{X}_f$. It then follows from **A1** and **A2** that $\tilde{\mathbf{u}}(k+1)$ is also feasible at time $k+1$, i.e. the feasibility of the problem at time $k+1$ is guaranteed by the feasibility at time k. Also from

$$J^*(k+1)=J_N(x(k+1),\mathbf{u}^*(k+1))\le J_N(x(k+1),\tilde{\mathbf{u}}(k+1)),$$

we have

$$\begin{aligned}&J^*(k+1)\le J_N(x(k+1),\tilde{\mathbf{u}}(k+1))=\\&J^*(k)-l(x(k),u^*(k|k))+l(x^*(k+N|k),x_f(x^*(k+N|k))\\&+V(Ax^*(k+N|k)+Bk_f(x^*(k+N|k)))-V(x^*(k+N|k)).\end{aligned}$$

The proof is now completed in view of **A.4**. ■

This theorem shows that if $\mathcal{X}_f$ is a feasible and invariant set for $x(k+1)=Ax(k)+Bk_f(x(k))$, MPC is stabilizing and its domain of attraction is the set of the initial state vectors which can be steerable to $\mathcal{X}_f$ in N steps or less. An interesting consequence is that the MPC can be globally stabilizing if $k_f(x)$ is found such that $\mathcal{X}_f=\mathbb{R}^n$. This is in fact possible if the unconstrained plant is neutrally stable and if constraints are imposed only on the input, i.e. $\mathcal{X}=\mathbb{R}^n$, as discussed on small gain control in the introduction.

3. A non-quadratic Lyapunov function for global stability

We first present a slight extension of the previous results [4], [5], in which the case where all poles are simple on the unit circle is considered first and then the general case for neutrally stable systems is handled via coordinate transformations. On the other hand, neutrally stable systems are directly dealt with in this paper.

Consider the following neutrally stable plant

$$x^+=Ax+B\mathrm{sat}(u) \quad (2)$$

where x^+ is the successor state (i.e. the state at the next sampling instant), $\mathrm{sat}(\cdot)$ is the usual saturation function as in [9], (A,B) is controllable and all the eigenvalues of A lie within and on the unit circle with those on the unit circle being simple. It then follows that there exists a positive definite matrix M_c satisfying

$$A^TM_cA-M_c\le 0.$$

Now the globally stabilizing small gain control is given by

$$u=-\kappa B^TM_cAx \quad (3)$$

where $\kappa\ (>0)$ satisfies

$$2\kappa B^TM_cB<I.$$

This control law is similar to those in [4], [5], [10]. It can then be shown that there exists a positive definite matrix M_q such that

$$(A-\kappa BB^TM_cA)^TM_q(A-\kappa BB^TM_cA)-M_q=-I.$$

The stability properties of the resulting closed-loop are given below.

Theorem 2: For the closed-loop system (2) and (3), there exists a Lyapunov function $W(x)$ such that

$$W(x) = W_q(x) + \lambda W_c(x) = x^T M_q x + \lambda (x^T M_c x)^{\frac{3}{2}} \quad (4)$$
$$W(x^+) - W(x) \leq -\|x\|^2$$

for some positive λ.

Proof: Note that the Lyapunov function (4) consists of a cubic term ($W_c(x)$) as well as a conventional quadratic term ($W_q(x)$). We first consider the difference of the quadratic term W_q along the trajectory, which is given by

$$W_q(x^+) - W_q(x) \leq -\|x\|^2 + 2a_1\|x\|u^T\sigma(u)$$

where $a_1 = \sigma_{\max}(A^T M_q B)$ with $\sigma_{\max}$ denoting the maximum singular value. Then, after some manipulations similar to those in [4], we obtain the difference of the cubic term W_c as follows:

$$W_c(x^+) - W_c(x) \leq -\frac{a_2}{\kappa}\|x\|u^T\sigma(u)$$

where $a_2 = \sqrt{\lambda_{\min}(M_c)}$ with $\lambda_{\min}$ denoting the minimum eigenvalue. From these differences of the quadratic and cubic components of the Lyapunov function, it is clear that if λ is set to

$$\lambda = \frac{2\kappa\sigma_{\max}(A_c^T M_q B)}{\sqrt{\lambda_{\min}(M_c)}},$$

then we have

$$\begin{aligned} W(x^+) - W(x) &= W_q(x^+) - W_q(x) + \lambda(W_c(x^+) - W_c(x)) \\ &\leq -\|x\|^2. \end{aligned}$$

This completes the proof. ■

4. A globally stabilizing MPC

On the basis of the discussions in sections 2 and 3, we derive a globally stabilizing MPC for the plant in (2) here. The key idea is to use a non-quadratic function of the form (4) as the terminal cost; since (4) is a global Lyapunov function, the resulting MPC can be globally stabilizing in view of **A4** of Theorem 1. We are now in a position to present our main theorem below.

Theorem 3: Consider the neutrally stable plant (2) and the following MPC law:

$$\begin{aligned} &J_N(x(k), \mathbf{u}(k)) \\ &= \sum_{i=0}^{N-1} l(x(k+i|k), u(k+i|k)) + V(x(k+N|k)) \end{aligned}$$

subject to

$$x(k+i+1|k) = Ax(k+i|k) + Bu(k+i|k), \; x(k|k) = x(k)$$
$$u(k+i|k) = \mathrm{sat}(u(k+i|k)), \quad i \in [0, N-1]$$

where

$$V(x(k+N|k)) = \epsilon W(x(k+N|k)),$$

and $l(x, u)$ and $W(x)$ are defined as in equations (1) and (4), respectively. Then, given any positive integer N, the closed-loop is globally asymptotically stable for some positive ϵ.

Proof: Note that this theorem holds if all the assumptions in **A.1**-**A.4** of Theorem 1 are satisfied for $\mathcal{X}_f = \mathcal{X} = \mathbb{R}^n$. Since assumptions **A.1**-**A.3** trivially hold, what remains is to find ϵ such that assumption **A.4** is satisfied. To this end, choose

$$k_f(x) = -\mathrm{sat}(\kappa B^T M_c A x),$$

and consider

$$\begin{aligned} l(x, k_f(x)) &\leq x^T Q x + \kappa^2 x^T A^T M_c B R B^T M_c A x \\ &\leq \lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A)\|x\|^2. \end{aligned}$$

Hence, if ϵ is chosen such that

$$\epsilon \geq \lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A),$$

then we have

$$\begin{aligned} &V(Ax^*(k+N|k) + Bk_f(x^*(k+N|k))) - V(x^*(k+N|k)) \\ &\leq -l(x^*(k+N|k), x_f(x^*(k+N|k)). \end{aligned}$$

This finally leads to

$$J^*(k+1) - J^*(k) \leq -l(x(k), u(k))$$

for all $x(k) \in \mathbb{R}^n$. ■

Remark 1: The proposed MPC is no longer a quadratic programming (QP) optimization problem. However, it is still convex, and can thus be dealt with effectively via various convex optimization solvers. For example, we employ an SQP(Sequential Quadratic Programming) algorithm for simulations of the next section.

Remark 2: In [7], infinite horizon MPC is needed in order to achieve global asymptotic stability. However, *any fixed finite* horizon can be used in the proposed MPC.

Remark 3: Recently some results have been presented on stability of nonlinear systems in the sense of ISS (Input to State Stability) or iISS (integral Input to State Stability). Among these is an interesting report proving that global stability is equivalent to iISS in discrete-time[11]. Hence the proposed MPC is actually integral input to state stabilizing, i.e. the state of the resulting closed-loop is guaranteed to be bounded even in the presence of an external disturbance provided that the disturbance has bounded energy. A consequence of this iISS property is that it will be possible to design a switching-based adaptive MPC for uncertain input-constrained neutrally stable systems. See e.g. [12] and [14] for switching-based adaptive control of nonlinear systems with iISS properties.

5. Simulation

To demonstrate the effectiveness of the proposed MPC scheme, we consider the following plant

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad -1 \leq u \leq 1.$$

Note that the unconstrained part of the system is neutrally stable with one integrator. For implementation, we employ an SQP algorithm in the optimization toolbox for Matlab. The MPC parameters used in the simulation are summarized below.

$$N = 3, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; R = 0.8,$$

$$M_c = \begin{bmatrix} 0.06 & 0.3 \\ 0.3 & 2 \end{bmatrix}, \; \kappa = \frac{0.95}{2\lambda_{\max}(B^T M_c B)},$$

$$\epsilon = \lambda_{\max}(Q + \kappa^2 A^T M_c B R B^T M_c A).$$

As shown in figures 1 and 2, the proposed MPC successfully stabilizes the neutrally stable plant while satisfying the saturation constraint.

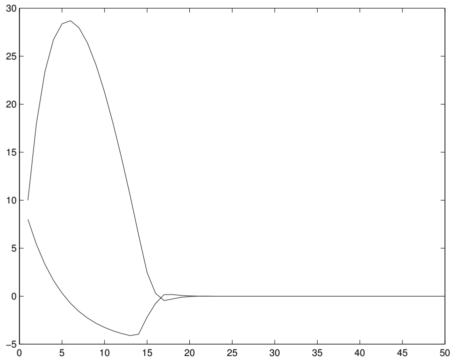

Fig. 1. State

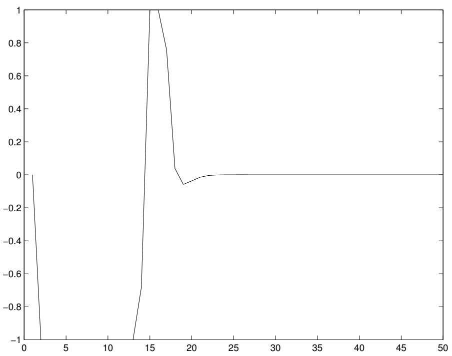

Fig. 2. Control input

6. Conclusion

In this paper, a finite horizon MPC is proposed, which globally stabilizes discrete-time neutrally stable linear systems subject to input constraints. The global stabilization is achieved by employing a non-quadratic function as the terminal cost, which consists of cubic as well as quadratic functions of the state. This is a discrete-time version of a recent work for continuous-time systems, and is to form a basis for switching-based adaptive MPC of input-constrained neutrally stable systems.

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