Angle and Position Control of Inverted Pendulum on a Cart Using Partial Feedback Linearization

Dong-Hae Yeom, and Jin-Young Choi

School of Electrical Engineering and Computer Science, Seoul National University, Seoul, Korea

(Tel : +82-2-872-7283; E-mail: dhyeom@neuro.snu.ac.kr)

Abstract : In this paper, we propose a controller for the position of a cart and the angle of a pendulum. To achieve both purposes simultaneously, we divide the system into the dominant subsystem and the dominated one after partial feedback linearization. The proposed controller is composed of a nonlinear controller stabilizing the dominant subsystem and a linear quadratic controller. Using the proposed controller, the controllable region is increased by the nonlinear control part and the control input minimized by the linear control part (LQR).

Keywords : Inverted pendulum, Partial feedback linearization, Dominant and dominated subsystem, Linear quadratic regulator

1. INTRODUCTION

An inverted pendulum on a cart is a typical nonlinear system with an unstable equilibrium point. This has been used a test bed to evaluate the performance of a controller [8]. In general, we design the controller based on the approximated linear model around the operating point in case nonlinear systems. But this linear controller is not valid any more when the system is out of operating region.

There are several approaches for the control of a nonlinear system. The feedback linearization through coordinate changes and nonlinear state feedback is a strong tool for nonlinear systems. Once a system is represented in a linear form, we can apply any conventional linear control method. Unfortunately the system of an inverted pendulum on a cart is not exact feedback linearizable and is partial feedback linearizable at the most. We can not apply back stepping to the system since the partially linearized system is not a triangular form [5].

Meanwhile we can use a switching control in case there is not single Lyapunov function or a system has different model according to different operating points. But the switching controller has a confirmed drawback entitled chattering which occurs on switching surface [2]. Intelligent schemes such as neural networks and fuzzy control of nonlinear systems have received much attention in recent times. However the performance of neural network controllers is much dependent on selected network structures and training data sets [1]. The fuzzy modeling does not exactly represent a nonlinear system since the modeling blends local linear model [6].

In this paper, we propose a controller for the position of a cart and the angle of a pendulum. After partial feedback linearization, we divide the system into the dominant subsystem and the dominated one through consideration of the system. The dominant subsystem consists of states determining the aspect of whole system. The first object of the proposed controller is to stabilize the dominant subsystem and to prevent the divergence of the dominated subsystem. Then the proposed controller stabilizes the dominated subsystem without disturbing the stability of the dominant subsystem. This controller is composed of a nonlinear controller related to dominant states and a linear quadratic controller. Doing this, the controllable region is increased by the nonlinear control part and the control input minimized by the linear control part (LQR).

This paper is organized as follows. Section 2 describes the derivation of mathematical model of the inverted pendulum on a cart from Newton's motion equations. Simplifying the system by partial feedback linearization and dividing the simplified system into the dominant subsystem and the dominated one are addressed in Section 3. Section 4 investigates the stability of the system when the proposed controller is employed. Finally, simulations and conclusions are addressed in Section 5 and 6 respectively.

2. MODELING OF INVERTED PENDULUM

In this section, we derive the mathematical model of an inverted pendulum on a cart [4]. To model the plants, we assume the followings

1) The mass of a pendulum is concentrated at the end of a rod.

2) A rod is a massless rigid body

3) The inertial moment of a pendulum with respect to the center of mass is equal to zero.

The diagram of an inverted pendulum on a cart is shown in the following Fig.1.

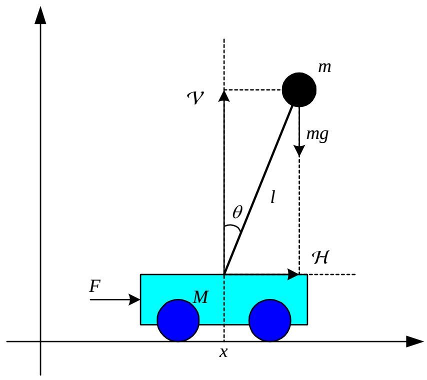

Fig. 1. The diagram of a inverted pendulum on a cart

Here m is the mass of an pendulum, M is the mass of a cart, l is the length of a rod, k is the friction coefficient between a cart and ground, I is the inertial moment of a pendulum with respect to the center of gravity, g is the acceleration of gravity, θ is the angle of a pendulum from the vertical line, x is the displacement of a cart, and F is the input force applying to a cart.

By Newton's law, the force acting on an inverted pendulum on a cart consists of a vertical component and a horizontal one.

$$m\frac{d^2}{dt^2}\left(x+l\sin\theta\right)=\mathcal{H},$$
$$m\frac{d^2}{dt^2}\left(l\cos\theta\right)+mg=\mathcal{V}. \tag{1}$$

The equations of the motion of a cart and a pendulum are

$$M\frac{d^2x}{dt^2}=u-\mathcal{H}-k\dot{x},$$
$$I\ddot{\theta}=\mathcal{V}l\sin\theta-\mathcal{H}l\cos\theta. \tag{2}$$

Eliminating $\mathcal{V}$ and $\mathcal{H}$ in equation (1) using equation (2) yields

$$I\ddot{\theta}=mgl\sin\theta-ml^2\ddot{\theta}-ml\ddot{x}\cos\theta,$$
$$M\ddot{x}=u-m\left(\ddot{x}+l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta\right)-k\dot{x}.$$

The above equations can be arranged with respect to $\ddot{x}$ and $\ddot{\theta}$.

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}=\frac{1}{\Lambda(\theta)}\begin{bmatrix}-ml\cos\theta & I+ml^2\\ M+m & -ml\cos\theta\end{bmatrix}\begin{bmatrix}mgl\sin\theta\\ u+ml\dot{\theta}^2\sin\theta-k\dot{x}\end{bmatrix},$$

where $\Lambda(\theta)=(I+ml^2)(M+m)-m^2l^2\cos^2\theta$.

Define the state variables to obtain the state space model.

$$x_1=x-r,$$
$$x_2=\dot{x},$$
$$x_3=\theta,$$
$$x_4=\dot{\theta},$$

where r means the reference position for a cart. According to assumptions, set the inertia moment of a pendulum (I) as 0 then the state space equation of an inverted pendulum on a cart is as follows

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=\dfrac{1}{M+m\sin^2x_3}\left\{-mg\sin x_3\cos x_3+mlx_4^2\sin x_3-kx_2\right\}+\dfrac{1}{M+m\sin^2x_3}u\\ \dot{x}_3=x_4\\ \dot{x}_4=\dfrac{1}{l(M+m\sin^2x_3)}\left\{-mlx_4^2\sin x_3\cos x_3+kx_2\cos x_3+(M+m)g\sin x_3\right\}\\ \qquad+\dfrac{-\cos x_3}{l(M+m\sin^2x_3)}u\end{cases}. \tag{3}$$

We define the position of a cart (x_1) and the angle of a pendulum (x_3) as outputs of the system.

3. PARTIAL FEEDBACK LINEARIZATION

In this section, we simplify the model obtained in Section 2 by partially feedback linearization, and show that the system consists of dominant subsystem and dominated one through consideration of simplified dynamics.

Equation (3) is represented as a control affine form.

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=f_2(x)+g_2(x)u\\ \dot{x}_3=x_4\\ \dot{x}_4=f_4(x)+g_4(x)u,\end{cases} \tag{4}$$

where

$$f_2=\frac{1}{M+m\sin^2x_3}\left\{-mg\sin x_3\cos x_3+mlx_4^2\sin x_3-kx_2\right\},$$
$$f_4=\frac{1}{l(M+m\sin^2x_3)}\left\{-mlx_4^2\sin x_3\cos x_3+kx_2\cos x_3+(M+m)g\sin x_3\right\}, \tag{5}$$
$$g_2=\frac{1}{M+m\sin^2x_3},$$
$$g_4=\frac{-\cos x_3}{l(M+m\sin^2x_3)},$$

with properties $f_2(0)=f_4(0)=0$, $g_2(0)\neq 0$, $g_4(0)\neq 0$.

Definition (partial feedback linearization[5]**)** Consider nonlinear single input system

$$\dot{x}=f(x)+g(x)u,\ \ x\in\mathbf{R}^n,\ u\in\mathbf{R},$$

in a neighborhood $U_{x_e}\subset\mathbf{R}^n$ of an equilibrium point x_e corresponding to $u=0$, i.e. $f(x_e)=0$. f and g are assumed to be smooth vector fields defined on $\mathbf{R}^n$ with $g(x_e)\neq 0$. The system is said to be locally partial state feedback linearizable with index $r\leq n$ if it is locally feedback equivalent to the partially linear system

$$\dot{\xi}=\chi(\xi,z),\quad \xi\in\mathbf{R}^{n-r},\ z\in\mathbf{R}^r,$$

$$\dot{z}=\begin{bmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & 0\end{bmatrix}z+\begin{bmatrix}0\\ 0\\ \vdots\\ 1\end{bmatrix}v.$$

□

According to the definition, we apply partial feedback linearization to the system (4) i.e. set input (u) as

$$u=\frac{1}{g_4(x)}\left[-f_4(x)+v\right].$$

Then the system is transformed as follows

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=f_2(x)+\dfrac{g_2(x)}{g_4(x)}\left[-f_4(x)+v\right]\\ \dot{x}_3=x_4\\ \dot{x}_4=v\end{cases}. \tag{6}$$

The second equation of (6) describes the acceleration of a cart. Rearranging this equation through (5) yields

$$\begin{aligned}\dot{x}_2&=\frac{1}{M+m\sin^2x_3}\left\{\frac{1}{\cos x_3}\left[(M+m)g\sin x_3-mg\sin x_3\cos^2x_3\right]\right\}-\frac{v}{\cos x_3}\\ &=\frac{1}{M+m\sin^2x_3}\left\{\frac{1}{\cos x_3}\left[(M+m)g\sin x_3-mg\sin x_3(1-\sin^2x_3)\right]\right\}-\frac{v}{\cos x_3}\\ &=\frac{1}{M+m\sin^2x_3}\left[\frac{1}{\cos x_3}\left(Mg\sin x_3+mg\sin^3x_3\right)\right]-\frac{v}{\cos x_3}\\ &=\frac{1}{M+m\sin^2x_3}\left[\frac{g\sin x_3}{\cos x_3}\left(M+m\sin^2x_3\right)\right]-\frac{v}{\cos x_3}\\ &=\frac{1}{\cos x_3}\left(g\sin x_3-v\right).\end{aligned}$$

Therefore we can simply represent the dynamics of the system with additional control input (v) as follows.

$$\begin{cases}\left.\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=\frac{1}{\cos x_3}\left(g\sin x_3-v\right)\end{aligned}\right\}\text{ part 1}\\ \left.\begin{aligned}\dot{x}_3&=x_4\\ \dot{x}_4&=v\end{aligned}\right\}\text{ part 2.}\end{cases} \tag{7}$$

Looking into the above system, the relationship between input (v) and output (x_3) is linear and the states of part 1 does not appear in part 2 explicitly, while part 1 depends on part 2 too much. If the state variables of part 2 approach 0, part 1 becomes a linear system,

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=gx_3-v\end{cases}. \tag{8}$$

Therefore part 2 is the dominant subsystem which determines the aspect of whole system, and part 1 is the dominated subsystem.

4. STABILTY ANALYSIS

In this section, we design a controller which stabilizes the system asymptotically. As you see in Section 3, the dynamics of the inverted pendulum on a cart is locally equivalent to (7). Hence we concentrate on the additional control input (v) and assume that the angle of the pendulum (x_3) is restricted within $(-\pi/2,\ \pi/2)$. If we set $v=-\alpha x_3-\beta x_4$, where $\alpha>0$ and $\beta>0$ are constants, part 2 in (7) is exponentially stable, but part 1 may diverse. If we set $v=-\gamma x_1-\delta x_2-\alpha x_3-\beta x_4$ with appropriate gain set $\{\gamma,\delta,\alpha,\beta\}$, the linearized system consisting of part 2 in (7) and (8) is exponentially stable also, but that is nothing but a linear controller.

We want to design a controller which stabilizes the dominant subsystem, prevents the divergence of the dominated subsystem, and finally stabilizes the whole system. Set the additional control input as follows

$$v=-\alpha x_3-\beta x_4+\rho\mu, \tag{9}$$

where $\mu=\gamma x_1+\delta x_2$ with constants $\gamma>0$, $\delta>0$. Here ρ needs to have such properties that it prevents the divergence of the dominated subsystem as long as the dominant subsystem is stabilized, and that it converges to an appropriate constant when the system becomes almost linear.

Theorem 1. For given α,β if there exist constant $a,\ b,\ c$ such that

$$M=\begin{bmatrix} a/2 & c/2 \\ c/2 & b/2 \end{bmatrix}>0,$$

$$N=\begin{bmatrix} \alpha c & (-a+\alpha b+\beta c)/2 \\ (-a+\alpha b+\beta c)/2 & \beta b-c \end{bmatrix}>0,$$

then using $\rho=\dfrac{x_{34}^T N x_{34}}{\left(\gamma|x_1|+\delta|x_2|+\varepsilon_1\right)\left(c|x_3|+b|x_4|+\varepsilon_2\right)}-\omega$, $\|x_{34}\|$ can be ultimately bounded sufficiently small , where $\omega=\exp\left[\dfrac{-(\gamma x_1+\delta x_2)^2}{\Delta_1^2}\right]\exp\left[\dfrac{-(cx_3+bx_4)^2}{\Delta_2^2}\right]$, $\varepsilon_1,\ \varepsilon_2\geq 1$, and $\Delta_1>>1$, $0<\Delta_2<<1$.

Proof.

Put a Lyapunov function candidate for x_3 and x_4 as follows

$V_{34}=x_{34}^T M x_{34}$, $x_{34}=\left[x_3\ x_4\right]^T$.

Derivative the function with respect to time along the trajectory

$$\begin{aligned}\dot V_{34}&=ax_3x_4+bx_4\left(-\alpha x_3-\beta x_4+\rho\mu\right)+cx_4^2+cx_3\left(-\alpha x_3-\beta x_4+\rho\mu\right)\\&=-\alpha c x_3^2-\left(\beta b-c\right)x_4^2-\left(-a+\alpha b+\beta c\right)x_3x_4+\rho\mu\left(cx_3+bx_4\right)\\&=-x_{34}^T N x_{34}+\rho\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)\\&=x_{34}^T N x_{34}\left[\frac{\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)}{\left(\gamma|x_1|+\delta|x_2|+\varepsilon_1\right)\left(c|x_3|+b|x_4|+\varepsilon_2\right)}-1\right]\\&\quad-\omega\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right).\end{aligned}$$

In case $(\gamma x_1+\delta x_2)(cx_3+bx_4)\geq 0$, $\dot V_{34}<0$. In case $\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)<0$, because $y\cdot\exp\left[-\dfrac{y^2}{\Delta^2}\right]$ has maximum value $\pm\sqrt{\dfrac{\Delta^2}{2e}}$ at $y=\pm\sqrt{\dfrac{\Delta^2}{2}}$, $\omega\left|\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)\right|\leq\dfrac{\Delta_1\Delta_2}{2e}$.

Hence $\dot V_{34}$ is bounded as follows

$$\dot V_{34}\leq x_{34}^T N x_{34}\left[\frac{-\left|\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)\right|}{\left(\gamma|x_1|+\delta|x_2|+\varepsilon_1\right)\left(c|x_3|+b|x_4|+\varepsilon_2\right)}-1\right]+\frac{\Delta_1\Delta_2}{2e}.$$

In order to $\dot V_{34}<0$

$$x_{34}^T N x_{34}>\frac{\Delta_1\Delta_2}{2e}>\frac{\Delta_1\Delta_2}{2e\left[1+\phi(x)\right]},$$

where $\phi(x)=\dfrac{\left|\left(\gamma x_1+\delta x_2\right)\left(cx_3+bx_4\right)\right|}{\left(\gamma|x_1|+\delta|x_2|+\varepsilon_1\right)\left(c|x_3|+b|x_4|+\varepsilon_2\right)}$ is restricted within $[0,1)$.

That means

$$\dot V_{34}<\left(\frac{\Delta_1\Delta_2}{2e}+\varepsilon\right)\left[-1-\phi(x)\right]+\frac{\Delta_1\Delta_2}{2e}=-\varepsilon-\phi(x)\left(\frac{\Delta_1\Delta_2}{2e}+\varepsilon\right)<0,$$

whenever $x_{34}^T N x_{34}>\dfrac{\Delta_1\Delta_2}{2e}$, where $\varepsilon>0$. We can assign $\Delta_1\Delta_2$ small arbitrarily, so $\|x_{34}\|$ can be bounded small sufficiently. Moreover there exist class $\mathcal{K}$ function, α_1 and α_2 and continuous positive definite function $\mathcal{W}$ such that

$$\alpha_1\left(\|x_{34}\|\right)\leq V_{34}\leq\alpha_2\left(\|x_{34}\|\right),$$

$$\dot V_{34}\leq-\mathcal{W}\left(x_{34}\right),\ \text{whenever}\ x_{34}^T N x_{34}>\frac{\Delta_1\Delta_2}{2e}.$$

Hence x_{34} is ultimately bounded [4].

Meanwhile the existence of positive definite matrices M, N can be shown by checking the leading principle minors of the matrices.

$$a>0,\ ab-c^2>0,\ \alpha c>0,\ \alpha c(\beta b-c)-(-a+\alpha b+\beta c)^2/4>0.$$

The set $\{\alpha,\beta,a,b,c\}$ which satisfies the above inequalities exists as follows

$\alpha>0$,

$\beta>0$,

$a>0$,

$$\frac{2\alpha a+a\beta^2}{2a^2}-\frac{1}{2}\sqrt{\frac{4\alpha\beta^2a^2+a^2\beta^4}{\alpha^4}}<b<\frac{2\alpha a+a\beta^2}{2a^2}+\frac{1}{2}\sqrt{\frac{4\alpha\beta^2a^2+a^2\beta^4}{\alpha^4}}$$

and

$$\frac{a\beta+\alpha\beta b}{4a+\beta^2}-2\sqrt{-\frac{\alpha a^2-2\alpha^2ab+\alpha^3b^2-\alpha a\beta^2 b}{\left(4a+\beta^2\right)^2}}<c$$

$$<\frac{a\beta+\alpha\beta b}{4a+\beta^2}+2\sqrt{-\frac{\alpha a^2-2\alpha^2ab+\alpha^3b^2-\alpha a\beta^2 b}{\left(4a+\beta^2\right)^2}}$$

, for example $\alpha=14$, $\beta=5$, $a=9$, $b=2$, $c=2$.

□

The theorem 1 implies that the states of the dominant subsystem can be bounded small arbitrary and that they never escape once the states come into the boundary. Then the system (7) becomes almost linear, or precisely speaking the system becomes a linear system with uncertainty.

We can divide the system (7) into the linear part and the uncertainty part as follows

$$\begin{cases}\dot x_1=x_2\\ \dot x_2=gx_3-\tilde u+\left(-gx_3+\tilde u+g\tan x_3-\dfrac{v}{\cos x_3}\right)\\ \dot x_3=x_4\\ \dot x_4=\tilde u+\left(-\tilde u+v\right)\end{cases},$$

in state space form

$$\dot x=Ax+B\tilde u+h(x), \tag{10}$$

where

$$A=\begin{bmatrix}0&1&0&0\\0&0&g&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\ B=\begin{bmatrix}0\\-1\\0\\1\end{bmatrix},\ h(x)=\begin{bmatrix}0\\-gx_3+\tilde{u}+g\tan x_3-\dfrac{v}{\cos x_3}\\0\\-\tilde{u}+v\end{bmatrix},$$

$\tilde{u}=-k^T x$, $k^T=\begin{bmatrix}\gamma & \delta & \alpha & \beta\end{bmatrix}$.

We investigate the robustness of the system (10) in the presence of some nonlinear perturbation $h(x)$. Here the pair (A,B) is controllable. The performance index to be minimized is

$$J=\int_0^\infty e^{2\eta t}\left[x^TQx+\tilde{u}^TR\tilde{u}\right]dt, \tag{11}$$

where the weighting matrices Q and R are positive definite and $\eta>0$ is a constant. The optimization of the performance index with the system $\dot{x}=Ax+B\tilde{u}$ yields the state feedback control

$$\tilde{u}=-k^Tx,\ k^T=R^{-1}B^TP, \tag{12}$$

where $P>0$ is the solution of the following algebraic Riccati equation

$$(A+\eta I)^T P+P(A+\eta I)-PBR^{-1}B^TP+Q=0. \tag{13}$$

The resulting closed loop system is given by

$$\dot{x}=\left(A-BR^{-1}B^TP\right)x+h(x).$$

If $h(x)=0$ the system is stable, moreover some bounds on $h(x)$ preserve the stability of the system.

Theorem 2. Let $D=2Q+(A+\eta I)^T P+P(A+\eta I)$ and let $\|\cdot\|_E$, $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ denote the Euclidean norm, maximum eigenvalue, and minimum eigenvalue respectively. If the uncertainty term $h(x)$ satisfies the condition

$$\frac{\|h(x)\|_E}{\|x\|_E}<\frac{\lambda_{\min}(D)}{2\lambda_{\max}(P)}+\frac{\eta\lambda_{\min}(P)}{\lambda_{\max}(P)}. \tag{14}$$

Then the system $\dot{x}=Ax+B\tilde{u}+h(x)$ is asymptotically stable.

Proof.

Choose the Lyapunov function as $V=x^TPx$. Derivative the function with respect to time along the trajectory,

$$\begin{aligned}\dot{V}&=\dot{x}^TPx+x^TP\dot{x}\\&=x^T\left[\left(A-BR^{-1}B^TP\right)^TP+P\left(A-BR^{-1}B^TP\right)\right]x+2h^TPx\\&=x^T\left[A^TP+PA-2PBR^{-1}B^TP\right]x+2h^TPx.\end{aligned}$$

Using algebraic Riccati equation and $D=PBR^{-1}B^TP+Q$ yields

$$\begin{aligned}\dot{V}&=x^T(-2\eta P-D)x+2h^TPx\\&=-x^TDx-2\eta x^TPx+2h^TPx.\end{aligned} \tag{15}$$

Meanwhile the uncertainty term satisfies following inequalities.

$$h^TPx\le\|h\|_E\|Px\|_E\le\lambda_{\max}(P)\|h\|_E\|x\|_E.$$

By the condition (14)

$$\begin{aligned}h^TPx&\le\left[\frac{\lambda_{\min}(D)}{2\lambda_{\max}(P)}+\frac{\eta\lambda_{\min}(P)}{\lambda_{\max}(P)}\right]\lambda_{\max}(P)\|x\|_E^2\\&=\left[\frac{1}{2}\lambda_{\min}(D)+\eta\lambda_{\min}(P)\right]\|x\|_E^2.\end{aligned} \tag{16}$$

Applying the above relation (16) to (15) yields

$$\begin{aligned}\dot{V}&\le-x^TDx-2\eta x^TPx+2\left[\frac{1}{2}\lambda_{\min}(D)+\eta\lambda_{\min}(P)\right]\|x\|_E^2\\&=-x^T\left[\left(D-\lambda_{\min}(D)I\right)+2\eta\left(P-\lambda_{\min}(P)I\right)\right]x.\end{aligned} \tag{17}$$

It is easy to see that, because $D-\lambda_{\min}(D)I$ and $P-\lambda_{\min}(P)I$ are positive semi-definite matrices, $\dot{V}\le 0$ [7] which proves that x is bounded, ρ and v are bounded as well, and finally $\dot{x}$ is bounded. We can conclude that $\dot{x}^T\tilde{D}x+x^T\tilde{D}\dot{x}^T$ is bounded also, where $\tilde{D}=\left(D-\lambda_{\min}(D)I\right)+2\eta\left(P-\lambda_{\min}(P)I\right)$. On the other hand, (17) implies that

$$\lim_{t\to\infty}\int_0^t x^T\tilde{D}x\,d\tau\le-\lim_{t\to\infty}\int_0^t\dot{V}(\tau)\,d\tau\le V(0)-\lim_{t\to\infty}V(t)<\infty.$$

That means $x^T\tilde{D}x$ is integrable. According to Barbalat's Lemma [3], because $\dot{x}^T\tilde{D}x+x^T\tilde{D}\dot{x}^T$ is bounded, $\lim_{t\to\infty}x^T\tilde{D}x=0$ which implies x converges to zero as $t\to\infty$.

□

5. NUMERICAL RESULTS

In this section, we simulate the inverted pendulum on a cart when the proposed controller in Section 4 is applied. We want to move the cart to the destination without throwing down the pendulum. Here the performance index is chosen as

$$J=\int_0^\infty e^{2t}\left(\frac{x^Tx}{2}+\frac{\tilde{u}^T\tilde{u}}{2}\right)dt,$$

i.e. $\eta=1$, $Q=I/2$, $R=1/2$ in (11).

Hence the solution of the algebraic Riccati equation (13) is

$$P=\begin{bmatrix}8.1849&6.6430&27.1625&8.6966\\6.6430&6.7704&29.2429&9.3842\\27.1625&29.2429&135.8963&43.7813\\8.6966&9.3842&43.7813&14.8082\end{bmatrix}$$

with $\lambda_{\min}(P)=0.2195$, $\lambda_{\max}(P)=161.9458$.

The linear quadratic state feedback gain (12) is

$$k^T=\begin{bmatrix}\gamma&\delta&\alpha&\beta\end{bmatrix}=\begin{bmatrix}2.0536&2.6138&14.5383&5.4240\end{bmatrix}.$$

The matrix D in Theorem 2 is

$$D=\begin{bmatrix}17.3698&21.4708&119.4260&44.5557\\21.4708&27.8268&151.9985&56.7079\\119.4260&151.9985&845.9543&315.4237\\44.5557&56.7079&315.4237&118.1789\end{bmatrix}$$

with $\lambda_{\min}(D)=0.500027$, $\lambda_{\max}(D)=1007.83$. Finally the additional control input (9) is obtained by setting $a=9$, $b=2$, $c=2$, $\varepsilon_1=\varepsilon_2=1$, $\Delta_1=0.1$, $\Delta_2=30$ as shown in Theorem 1.

$$\begin{aligned}v=&-14.5383x_3-5.4240x_4+\left(2.0536x_1+2.6138x_2\right)\\&\times\left\{\frac{29.0766x_3^2+8.848x_4^2+30.9246x_3x_4}{\left(2.0536|x_1|+2.6138|x_2|+1\right)\left(2|x_3|+2|x_4|+1\right)}\right.\\&\left.-\exp\left[-\frac{\left(2.0536x_1+2.6138x_2\right)^2}{0.01}\right]\exp\left[-\frac{\left(2x_3+2x_4\right)^2}{900}\right]\right\}.\end{aligned}$$

Scenario 1 : The initial angle of the pendulum is -10 degree and the destination of the cart is 1m.

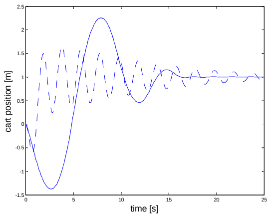

Fig. 2.

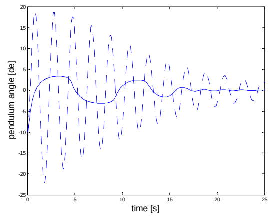

Fig. 3.

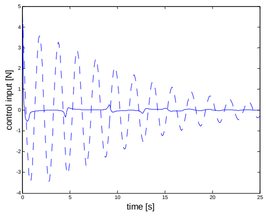

Fig. 4.

The position of the cart, the angle of the pendulum, and the additional control input are addressed at figure 2, 3, 4 respectively. The dotted line represents the situation only when the linear quadratic state feedback control, $v = -\gamma x_1 - \delta x_2 - \alpha x_3 - \beta x_4$ is applied.

Scenario 2 : The initial angle of the pendulum is 10 degree and the destination of the cart is 1m.

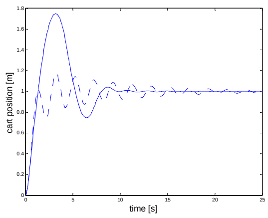

Fig. 5.

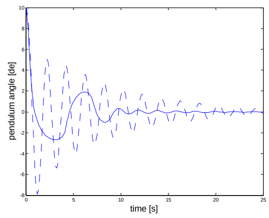

Fig. 6.

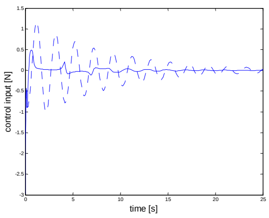

Fig. 7.

The position of the cart, the angle of the pendulum, and the additional control input are addressed at figure 5, 6, 7 respectively. The dotted line represents the situation only when the linear quadratic state feedback control is applied.

Even though the displacement of the cart is large relatively, the proposed controller reduces the settling time, the displacement of the pendulum, and the control effort remarkably as you see at figure 4 and 7. Moreover the controller extends the controllable region.

Figure 8 shows that the condition (14) in Theorem 2 is preserved since $\| h(x) \|_E / \| x \|_E$ converges to zero according as $\| x \|$ approaches zero as $t \to \infty$, where the dotted line represents the constant value of the right hand side of (14).

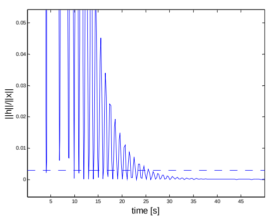

Fig. 8.

6. CONCLUSION

An inverted pendulum on a cart is a typical nonlinear system with an unstable equilibrium point. In case we intend to control both the position of the cart and the angle of the pendulum, the exact feedback linearization can not be applied directly since the dynamics of the system is entangled. In this paper, using partial feedback linearization we divided the system into the dominant subsystem and the dominated one. We have proposed a controller composed of the nonlinear controller for the dominant subsystem and the linear quadratic controller. The proposed controller has merits such as increasing the controllable region, minimizing the control effort, and reducing the settling time.

Reference

[1] C. W. Anderson, “Learning to Control an Inverted Pendulum Using Neural Networks”, *IEEE Control System Magagine*, Vol.9, pp. 31-37, April 1989.

[2] K. J. Astrom and K. Furuta, “Swinging Up a Pendulum by Energy Control ”, *Proceedings of 13th IFAC*, Vol. E, pp. 37-42, 1996.

[3] P. A. Ioannou and Jing Sun, “Robust Adaptive Control”, *Prentice Hall*, 1996.

[4] H. K. Khalil, “Nonlinear Systems”, *Prentice Hall*, 2002.

[5] Riccardo Marino and Patrizio Tomei, “Nonlinear Control Design”, *Prentice Hall*, 1995.

[6] P. P. Mohanlal and M. R. Kaimal, “Exact Fuzzy Modeling and Optimal Control of the Inverted Pendulum on Cart”, *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 3255-3260, December 2002.

[7] R. V. Patel and M. Toda, “Robustness of Linear Quadratic State Feedback Design in the Presence of System Uncertainty”, *IEEE Transactions on Automatic Control*, Vol. AC-22, No. 6, pp. 945-949, December 1977.

[8] Wei Zhong and Helmut Rock, “Energy and Passivity Based Control of the Double Inverted Pendulum on a Cart”, *Proceedings of the 2001 IEEE International Conference on Control Applications*, pp. 896-901, September 2001.