Robust adaptive fuzzy controller for an inverted pendulum

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Abstract: This paper proposes an indirect adaptive fuzzy controller for general SISO nonlinear systems. No a priori information on bounding constants of uncertainties including reconstruction errors and optimal fuzzy parameters is needed. The control law and the update laws for fuzzy rule structure and estimates of fuzzy parameters and bounding constants are determined so that the Lyapunov stability of the whole closed loop system is guaranteed. The computer simulation results for an inverted pendulum system show the performance of the proposed robust adaptive fuzzy controller.

Keywords: Fuzzy system, Universal approximation theorem, Inverted pendulum, Robust adaptive fuzzy controller

1. Introduction

Fuzzy system has been successfully applied to many control problems because it needs no accurate mathematical models of the system under control and it can be cooperate with human experts knowledge. Recently, it is shown that fuzzy systems, as well as neural networks, can approximate certain classes of functions to a given accuracy. Wang[1], [2] proved that the fuzzy systems are universal approximators and furthermore the output of the system can be represented by a linear combination of the so-called fuzzy basis functions. Based on this property many researchers presented an adaptive fuzzy control architecture for nonlinear systems. Comparing with conventional adaptive control, the key advantage of the control scheme using fuzzy systems is not need linear parameterization condition.

However, since the fuzzy descriptions are imprecise and may be insufficient to achieve the desired accuracy, the approximation error introduced into the feedback loop makes it difficult to guarantee the stability of the closed-loop control system[3]. This problem was solved in [4] [5] by sliding mode control which is introduced to estimation of the reconstruction error bound, but the discontinuous control input is generated due to this estimate. In general, such discontinuous adaptive control schemes are avoided since it is well known that they not only create problems of existence and uniqueness for solutions but also are known to display chattering phenomena and to excite high frequency unmodeled dynamics.

Another problem is that for tracking error to be bounded by a given constant, bounds on the unknown plant dynamics must be known. Generally, this calculation may require an exact model of the plant, which defeats the purpose of using a model free technique.

The purpose of this paper is to develop an indirect robust adaptive control algorithm against the reconstruction errors using fuzzy systems for single-input single-output(SISO) nonlinear dynamical system like the inverted pendulum system. Expanding the adaptive bounding technique to the dynamic system with state-dependent input gain, we propose a smooth control input with no chattering phenomena. An explicit linear parameterization of the uncertainties in the dynamics for the system is either unknown and impossible to determine. The proposed controller guarantees that tracking error converges in the small the neighborhood of zero under less restrictive assumptions and that the states and estimated parameters are all bounded. In the special case of the scheme, it is also shown that the tracking error exponentially converges to zero even though the approximation errors exist. Finally, the features of the proposed control approach are verified by the simulation of an inverted pendulum.

The remainder of the paper is organized as follows. Section 2 is devoted to the basic aspects of an adaptive fuzzy system. A new adaptive fuzzy sliding mode control approach is presented in section 3. An illustrative example is given in section 4. Finally, section 5 concludes the paper.

2. Indirect adaptive fuzzy control

In this section, we first set up the control objectives, and then show how to design an adaptive controller based on the fuzzy system to achieve these control objectives.

Consider the nth-order nonlinear system of the form

$$x^{(n)} = f(x, \dot{x}, \cdots, x^{(n-1)}) + g(x, \dot{x}, \cdots, x^{(n-1)})u \quad (1)$$
$$y = x$$

where f and g are unknown continuous smooth nonlinear functions, $u \in R$ and $y \in R$ are the input and output of the system, respectively, and $x \in R^n$ is the state vector of the system that is assumed to be available for measurement. The control objective is to determine a feedback control input u in order to force the output y to track a given bounded reference signal y_m and an adaptation law for adjusting the parameter vector θ, under the constraint that all signals involved must be bounded.

2.1 Fuzzy Model Identification

A fuzzy system consists of three part: fuzzifier, fuzzy rules and defuzzifier.

The fuzzifier maps an input point in the input space $U \in R^n$ to a fuzzy set A_x in the input space. Fuzzy sets are characterized by their membership functions $\mu(\cdot)$s. The inference engine performs a mapping from fuzzy sets in the input space to fuzzy sets in the output space $V \in R^m$ based on fuzzy rules.

A fuzzy rule base consists of a set of fuzzy IF-THEN rules as follows

R_{gj}: *if* $x_1 = G_{1j}$ and $x_2 = G_{2j}$ and ... and $x_n = G_{nj}$ *then* $\hat{g}(x \mid \theta_g) = \theta_g^T \xi(x)$

R_{fj}: *if* $x_1 = F_{1j}$ and $x_2 = F_{2j}$ and ... and $x_n = F_{nj}$ *then* $\hat{f}(x \mid \theta_f) = \theta_f^T \xi(x)$

where N is the total number of rules, F_{ij} and G_{ij} are the linguistic variables and $\theta_f = (\theta_{f1}, \theta_{f2}, \ldots, \theta_{fN})^T$, $\theta_g = (\theta_{g1}, \theta_{g2}, \ldots, \theta_{gN})^T$ are called the parameter vectors of f and g.

ξ_f and ξ_g are called fuzzy basis function vectors and defined by following forms.

$$\begin{aligned} \xi_f(x) &= (\xi_{f1}(x)\ \xi_{f2}(x), \ldots, \xi_{fN}(x))^T \\ \xi_g(x) &= (\xi_{g1}(x)\ \xi_{g2}(x), \ldots, \xi_{gN}(x))^T \end{aligned} \tag{2}$$

$$\begin{aligned} \xi_{fj}(x) &\triangleq \frac{\Pi_{i=1}^n \mu_{F_{ij}}(x_i)}{\Sigma_{j=1}^N \Pi_{i=1}^n \mu_{F_{ij}}(x_i)} \\ \xi_{gj}(x) &\triangleq \frac{\Pi_{i=1}^n \mu_{G_{ij}}(x_i)}{\Sigma_{j=1}^N \Pi_{i=1}^n \mu_{G_{ij}}(x_i)} \end{aligned} \tag{3}$$

The estimates $\hat{f}(x \mid \theta_f)$, $\hat{g}(x \mid \theta_g)$ of nonlinear function of a fuzzy system with center-average defuzzifier, product inference and singleton fuzzifier are of the following forms

$$\begin{aligned} \hat{f}(x \mid \theta_f) &= \frac{\sum_{j=1}^N \theta_{fj}\mu_{R_{fj}}(x)}{\sum_{j=1}^N \mu_{R_{fj}}(x)} \\ \hat{g}(x \mid \theta_g) &= \frac{\sum_{j=1}^N \theta_{gj}\mu_{R_{gj}}(x)}{\sum_{j=1}^N \mu_{R_{gj}}(x)} \end{aligned} \tag{4}$$

Then nonlinear function $\hat{f}(x)$, $\hat{g}(x)$ can be written as

$$\begin{aligned} \hat{f}(x|\theta_f) &= \theta_f^T \xi_f(x) \\ \hat{g}(x|\theta_g) &= \theta_g^T \xi_g(x) \end{aligned} \tag{5}$$

2.2 Adaptive law for parameter estimation

First, to find the adaptive laws of parameter vectors θ_f, θ_g of the fuzzy system $f(x), g(x)$, define the minimum approximation error.

$$w = (f(x) - \hat{f}(x|\theta_f^*)) + (g(x) - \hat{g}(x|\theta_g^*))\,u_{eq}^c \tag{6}$$

where θ_f^*, θ_g^* are called the optimal parameter vectors and are artificial quantities required only for analytical purposes. Typically, θ_f^*, θ_g^* are chosen as the values of θ_f, θ_g, respectively that minimize the reconstruction errors, i.e.,

$$\begin{aligned} \theta_f^* &= \min_{\theta_f \in \Omega_f}[\max_{x \in U_x} |f(x) - \hat{f}(x|\theta_f)|] \\ \theta_g^* &= \min_{\theta_g \in \Omega_g}[\max_{x \in U_x} |g(x) - \hat{g}(x|\theta_g)|] \end{aligned} \tag{7}$$

where Ω_f, Ω_g are constraint sets for θ_f and θ_g, respectively, specified by the designer.

$$\begin{aligned} \Omega_f &= \{\theta_f \mid |\theta_f| \le M_f\} \\ \Omega_g &= \{\theta_g \mid |\theta_g| \le M_g\} \end{aligned}$$

Let tracking error be defined as $e = y_d - y$

$$\boldsymbol{e} = (e, \dot{e}, \cdots, e^{(n-1)})^T \tag{8}$$

and choose the control law as

$$u_{eq} = \frac{1}{\hat{g}(x|\theta_g)}(-\hat{f}(x|\theta_g) + y_d^{(n)} + \boldsymbol{k}^T \boldsymbol{e}) \tag{9}$$

where u_{eq} is the so-called certainty equivalent control in order to compensate the unknown nonlinear term of the controlled system and stabilize the overall feedback control system. Let vector $\boldsymbol{k}$ be such that all roots of the polynomial $s^n + k_1 s^{n-1} + \cdots + k_n = 0$ are in the open left-half complex plane.

Applying (9) to (1) and after some straightforward manipulation, we obtain the error equation

$$\dot{\boldsymbol{e}} = \Lambda \boldsymbol{e} + b_c\{(\hat{f}(x) - f(x)) + (\hat{g}(x) - g(x))u_{eq}\} \tag{10}$$

where

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_n & -k_{n-1} & \cdots & \cdots & -k_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{bmatrix}.$$

Since Λ is a stable matrix, we know that there exists a unique positive symmetric $n \times n$ matrix P which satisfies the following Lyapunov equation

$$\Lambda^T P + P\Lambda = -Q \tag{11}$$

where Q is an arbitrary $n \times n$ positive definite matrix.

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\left\{\boldsymbol{e}^T P \boldsymbol{e} + \frac{1}{\gamma_1}\Phi_f^T \Phi_f + \frac{1}{\gamma_2}\Phi_g^T \Phi_g\right\} \tag{12}$$

where γ_1 and γ_2 are positive constants.

Differentiating V along the solution (10), we have

$$\begin{aligned} \dot{V} &= -\frac{1}{2}\, e^T Q e - \frac{1}{\gamma_1}\, \Phi_f^T \dot{\theta}_f - \frac{1}{\gamma_2}\, \Phi_g^T \dot{\theta}_g \\ &= -\frac{1}{2}\, e^T Q e - \frac{1}{\gamma_1}\, \Phi_f^T [\, \dot{\theta}_f - \gamma_1 e^T P b \xi_f] \\ &\quad -\frac{1}{\gamma_2}\, \Phi_g^T [\, \dot{\theta}_g - \gamma_2 e^T P b \xi_g u_{eq}] \end{aligned} \tag{13}$$

where we used (11) and $\dot{\Phi}_f = -\dot{\theta}_f$, $\dot{\Phi}_g = -\dot{\theta}_g$.

If we choose the adaptation law

$$\begin{aligned} \dot{\theta}_f &= \gamma_1 e^T P b\, \xi_f(x) \\ \dot{\theta}_g &= \gamma_2 e^T P b\, \xi_g(x) u_{eq} \end{aligned} \tag{14}$$

then from (13) we have the following derivative of the Lyapunov function.

$$\dot{V} \le -\frac{1}{2}\, e^T Q e + e^T P b w \tag{15}$$

This is the best we can hope to get because the term $e^T P b w$ is of the order of the minimum approximation error. If $w = 0$, then we have $\dot{V} \le 0$. Because the fuzzy systems are universal approximators, we can hope that the minimum approximation error w should be small, if not equal to zero, provided that we sufficiently increase the number of rules of the fuzzy system.

3. Robust adaptive fuzzy control

This type of adaptive fuzzy controller proposed by Wang has some disadvantages in the practical applications. To use this controller we need to know the some fixed bounds. In general, it is not easy to determine these bounds for the unknown system.

we propose the robust adaptive controller guarantees that tracking error converges in the small the neighborhood of zero under less restrictive assumptions and that the states and estimated parameters are all bounded. In the special case of the scheme, it is also shown that the tracking error exponentially converges to zero even though the approximation errors exist.

Consider the nth-order nonlinear system of the form

$$\begin{aligned} x^{(n)} &= f(x, \dot{x}, \cdots, x^{(n-1)}) + g(x, \dot{x}, \cdots, x^{(n-1)})\, u \\ y &= x \end{aligned} \tag{16}$$

In many practical applications, the control design has rough estimates of any unknown nonlinearities. To incorporate any such a priori information into our control design, the unknown functions f and g are expressed as

$$\begin{aligned} f(x) &= \hat{f}(x|\theta_f) + \delta_f(x) \\ g(x) &= \hat{g}(x|\theta_g) + \delta_g(x) \end{aligned} \tag{17}$$

where $\hat{f}(x|\theta_f)$ and $\hat{g}(x|\theta_g)$ are some estimates of f and g, respectively and $\delta_f(x)$ and $\delta_g(x)$ are unknown functions representing the system uncertainty.

Let the control law be chosen as

$$u = u_{eq} + u_r \tag{18}$$

where u_{eq} is the certainty equivalent controller given by (9) and u_r is the bounding controller and is needed to improve robustness with respect to the bounding uncertainty. We choose u_r as follows

$$u_r = \frac{\beta}{\hat{g}(x|\theta_g)} \tag{19}$$

$$\beta = \frac{\Psi(t)^T s}{1 - \Psi_g s_g / |\hat{g}|} \tag{20}$$

$$s = \begin{bmatrix} s_f sgn(z) \\ s_g |u_{eq}| sgn(z) \end{bmatrix} \tag{21}$$

$$z = e^T P b \tag{22}$$

where $\Psi = [\, \Psi_f \quad \Psi_g]^T$ is an estimation for the bounding constants.

To design the robust adaptive fuzzy controller we choose the following Lyapunov function

$$\begin{aligned} V &= \frac{1}{2} \Big\{ e^T P e + \frac{1}{\gamma_f}\, \Phi_f^T \Phi_f + \frac{1}{\gamma_g}\, \Phi_g^T \Phi_g \\ &\quad + \frac{1}{\gamma_\phi}\, \Psi^T \Psi \Big\} \end{aligned} \tag{24}$$

where P is a positive definite symmetric matrix which satisfies Lyapunov equation of (11) and γ_f, γ_g and γ_ϕ are positive adaptation gains.

By using (10), (11) and (18) the time derivative of V satisfies

$$\begin{aligned} \dot{V} &= -\frac{1}{2}\, e^T Q e - \frac{1}{\gamma_f}\, \Phi_f^T \dot{\theta}_f - \frac{1}{\gamma_g}\, \Phi_g^T \dot{\theta}_g \\ &\quad + \frac{1}{\gamma_\phi}\, \Psi^T \dot{\Psi} \\ &= -\frac{1}{2}\, e^T Q e - \frac{1}{\gamma_f}\, \Phi_f^T [\, \dot{\theta}_f - \gamma_f e^T P b \xi_f] \\ &\quad - \frac{1}{\gamma_g}\, \Phi_g^T [\, \dot{\theta}_g - \gamma_g e^T P b \xi_g u_{eq}] \\ &\quad + e^T P b \delta_f + e^T P b \delta_g u - e^T P b \beta + \frac{1}{\gamma_\phi}\, \Psi^T \dot{\Psi} \end{aligned} \tag{25}$$

Then, the adaptive laws for θ_f and θ_g are chosen as (11), we have

$$\dot{V} = -\frac{1}{2}\, e^T Q e + \Lambda \tag{26}$$

where

$$\begin{aligned} \Lambda &= e^T P b \delta_f + e^T P b \delta_g u - e^T P b \beta \\ &\quad + \frac{1}{\gamma_\phi}\, \Psi^T \dot{\Psi} \end{aligned} \tag{27}$$

Using similar method by Polycarpou and Mears[6] and Corless and Leitmann[7] we know that (27) is zero.[refer to reference [8]]

Then,

$$\dot{V} \le -\frac{1}{2}\, e^T Q e < 0 \tag{27}$$

From (27), we can prove the uniform ultimate boundness.

4. Computer Simulation

To illustrate the control procedure and performance we apply our robust fuzzy controller to control the inverted pendulum to a sinewave trajectory. Fig. 1 shows the inverted pendulum system. The dymanics of the system can be derived using the Euler-Lagrange method

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g\sin x_1 - \frac{mlx_2^2\cos x_1 \sin x_1}{m_c+m} + \frac{\cos x_1}{m_c+m}u}{D} \quad (28)$$

$$D = l\left(\frac{4}{3} - \frac{m\cos^2 x_1}{m_c+m}\right) \quad (29)$$

where x_1 and x_2 are the angular position and velocity of the pole, u is the control input torque applied to cart, m_c is the mass of cart, m is the mass of pole, l is the half-length of pole. We choose the parameters of the inverted pendulum system in table 1.

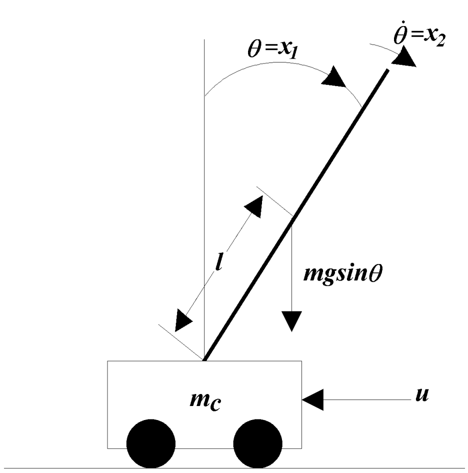

Fig. 1 The inverted pendulum system

Table 1 The parameter of the inverted pendulum

	Symbol	Value
acceleration of gravity	g	9.8 m/s2
mass of cart	mc	1 kg
mass of pole	m	0.1 kg
length of pole	l	0.5 m

The membership functions for x_1 and x_2 are chosen as a Gaussian-shaped form as follows

$$\mu_{F_i^1}(x_i) = \exp\left[-\left(\frac{x_i + \pi/6}{\pi/24}\right)^2\right]$$
$$\mu_{F_i^2}(x_i) = \exp\left[-\left(\frac{x_i + \pi/12}{\pi/24}\right)^2\right]$$
$$\mu_{F_i^3}(x_i) = \exp\left[-\left(\frac{x_i}{\pi/24}\right)^2\right] \quad (29)$$
$$\mu_{F_i^4}(x_i) = \exp\left[-\left(\frac{x_i - \pi/12}{\pi/24}\right)^2\right]$$
$$\mu_{F_i^5}(x_i) = \exp\left[-\left(\frac{x_i - \pi/6}{\pi/24}\right)^2\right] \qquad i = 1, 2$$

we also choose the reference signal $y_m = \frac{\pi}{30}\sin(t)$. The design parameters are specified as follows. The initial state is $x(0) = (-0.05, 0)^T$ and let k_1=2, k_2 =1 and $Q = 5I_{2\times 2}$, then we have Lyapunov equation (11) and obtain

$$P = \begin{bmatrix} 7.5 & 2.5 \\ 2.5 & 2.5 \end{bmatrix}$$

which is positive definite with $\lambda_{\min}(P) = 1.4645$.

Fig. 2 shows the trajectories of system output, desired output and the response of tracking error is allustrated Fig 2. From the results, it can be inferred that the system output tracks the desired output well by the proposed robust adaptive fuzzy controller.

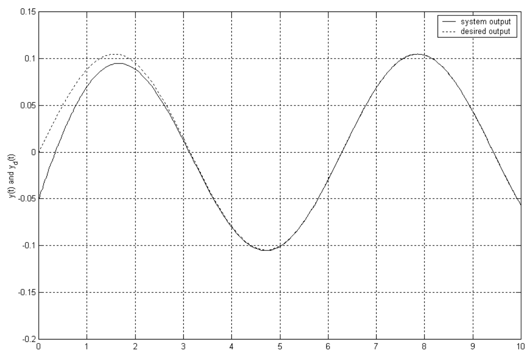

Fig. 2 system output and desired output

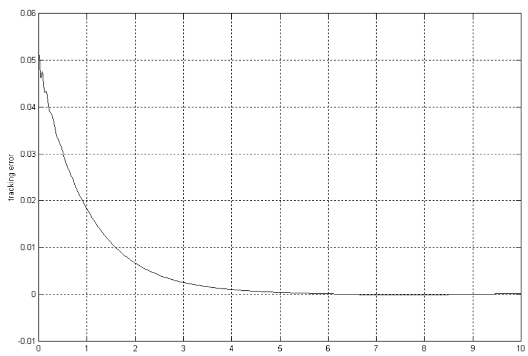

Fig. 3 the response of tracking error

5. Conclusions

In this paper we apply fuzzy systems to model the uncertain or ill-defined feedback linearizable nonlinear plant. Adaptive laws are developed to adjust the uncertain bounding constants and consequent parameters of the fuzzy rules. The control input consists of the certainty equivalent control and the robust control term which adaptively compensate the reconstruction errors. Adaptive laws and control input are established to stabilize the closed-loop system in the Lyapunov sense. The developed controller can guarantee that the tracking error converge to zero and that all signals involved are unfirmly bounded. Simulations for the inverted pendulum system demonstrate that the proposed control architecture provides good tracking performance.

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